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INVITED PAPER

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Chains of Life: Turing, *Lebensform*, and the Emergence of Wittgenstein's Later Style

Abstract

This essay accounts for the notion of *Lebensform* by assigning it a *logical* role in Wittgenstein's later philosophy. Wittgenstein's additions of the notion to his manuscripts of the *PI* occurred during the initial drafting of the book 1936-7, after he abandoned his effort to revise *The Brown Book*. It is argued that this constituted a substantive step forward in his attitude toward the notion of simplicity as it figures within the notion of logical analysis. Next, a reconstruction of his later remarks on *Lebensformen* is offered which factors in his reading of Alan Turing's "On computable numbers, with an application to the *Entscheidungsproblem*" (1936/7), as well as his discussions with Turing 1937-1939. An interpretation of the five occurrences of *Lebensform* in the *PI* is then given in terms of a logical "regression" to *Lebensform* as a fundamental notion. This regression characterizes Wittgenstein's mature answer to the question, "What is the nature of the logical?"

1. Introduction

Lebensform is an important gauge of the development of Wittgenstein's thought. It represents, from a regressive point of view, the place he finally chased down *to*. In foundational matters such as these, it is often the simplest, naïve perspective that comes

last in a thinker's (or a tradition's) evolution, the working through of how to get back home to the primordial basis, the extraction of fundamentals from a jungle of technical accretions. I believe that *Lebensform* functions as part of such an extraction in Wittgenstein's later writings. *Lebensform* plays, in this sense, a logical role. I do not mean by this that it is a "technical" term only, or that our lives are only logical. Rather, the notion of *Lebensform* is *recovered*, its role clarified in thinking through the nature of the logical, which I shall treat, following Wittgenstein, very broadly.

That the excavation of ultimate (logical) starting points often comes late in the chronological development of a thinker is no accident: it mirrors certain necessities of the regressive method. In the foundations of logic, the plea for evidence, for "common sense", for "simplicity", recurs thematically and repeatedly, but most pointedly at the end. The recognition that a certain conceptual residue must be accepted irreducibly at the first step – and how this is so – tends to come last, and is often the most difficult step.¹ For, in the end, the measure of a reduction's significance must lie in the fruitfulness of the resulting restructuring of our thought.

In this essay I will be joining Wittgenstein's philosophy to certain key elements and problematics of this logical tradition, as I think he himself did. To re-consider *Lebensformen* in this way, as an end-product of the regressive method, I impose a chronological development on his thought. I make much of the fact that *Lebensform* entered at a crucial juncture, the period 1936-7. This was the time when he generated the first two drafts of *Philosophical Investigations* (hereafter "PI") and broke through to his later, mature style of writing.²

Why does *Lebensform* enter his writing then? I propose one answer (there may be others). During this period – we are not sure exactly when, but it could have been any time between the spring of 1936 and the spring of 1937 – Wittgenstein came into

¹ Cf. Floyd 2013b on the history of a variety of modern rigorizations following this pattern; Russell 1903/1938, Preface (paragraph 2) comments on this very point.

² I am not the first to isolate these years as a turning point. Cf. Pichler 2004, Chapter 4.3, Schulte 2013, and Stern 2017.

possession of Alan Turing's epochal paper "On computable numbers: with an application to the *Entscheidungsproblem*" (hereafter "Turing 1936/7").³ In the summer of 1937 he met with Turing and discussed the paper with him, incorporating responses into his manuscripts that fall, when he completed the so-called *Frühversion*, or "early version" of PI.⁴ If we take this chronological story seriously, and conjecture what reading this paper may have done for Wittgenstein's conception of his great life-long question, "What is the nature of the logical?", we can see how and why it might be that *Lebensformen* came to play a certain role in his later works.

What follows uses a logical construction to construe the narrative; the latter is what is philosophically most important. Using the perspective framed in sections 2 and 3 of the paper, I argue that the *philosophical* impact these two thinkers had on each other was significant and mutual, and that this offers us a novel perspective on both. Taking my reconstruction of the logical situation into account, I turn in section 4 to a reconstruction of the five occurrences of *Lebensform* in PI. Like a falling barometer, the overarching sequence of passages presses ever more clearly toward the foundations of logic and philosophy in general. In the course of the journey, our very notion of the simple, the ordinary, of "common sense" itself, is reworked and recovered anew. *Lebensform* becomes more and more imbricated in everyday procedures, highlighting their ubiquity.

Elsewhere I have argued that it is likely that Turing was aware of *The Blue Book* and/or *The Brown Book* while still an undergraduate, and that exposure to Wittgenstein's philosophy in

³ On February 11, 1937 AT wrote to his mother that he had "already" sent off a copy of the paper to Wittgenstein (AMT/K/1/54, Turing Digital Archive (<http://www.turingarchive.org/browse.php/K/1/54>)). He placed Wittgenstein second on the list (after Littlewood) of those whom he had already contacted outside of King's College. The manuscript had been given to Newman by Turing in April 1936, the very month that Church finished his own proof of the impossibility of a decision procedure for logic. Newman did not read it until mid-May, thinking at first that it was too simple-minded. Only after reading Church 1936 did he see the magnitude of what Turing had done. Turing submitted his paper 28 May 1936, continuing work over that summer in Cambridge. By 28 August 1936 he had added an appendix showing that his definition of "computable" is extensionally equivalent to Church's notion of "effectively calculable" (described by a formula in Church's lambda-calculus). Cf. Hodges 1983/2012, Chapter 3.

⁴ For all these preliminary versions of the PI, I rely on Wittgenstein 2001.

1932-35 left its imprint, not only on the specific argumentation Turing gives in “On computable numbers”, but also on several subsequent papers he wrote, including one with explicit expressions of indebtedness to Wittgenstein.⁵ In this essay I broach the other, more difficult Turing-to-Wittgenstein direction of influence, embedding the notion of *Lebensform* in a conception of logic that belongs both to “On computable numbers” and to Wittgenstein’s later philosophical method.

The argument will be structured in the following way. The notion on which I rely, in imposing a chronological frame, is that of *simplicity*. This is a cousin of the idea of the everyday or the ordinary, and it is present throughout Wittgenstein’s career. We may schematically (and not terribly controversially) divide Wittgenstein’s development into a series of phases orbiting around his treatment of this notion.

In the *Tractatus*, simples are *absolute* ideals, absolute “undefinables”, objects reflected in what we say, and systematized in logic as it bottoms out in a well-founded, comprehensive analysis of what there is to be said, true or false. Logic is presented schematically, formally and operationally, according to a certain vision. At least ideally, logic is envisioned as representable from bottom to top, relying on the totality of elementary propositions, which are mutually independent. Of course, the approach to framing any such analysis would be guided by a more synthetic or holistic appreciation of interconnections among sentences appreciated in our applications of logic in everyday life.⁶ In particular, the variables Wittgenstein devises for “form series” [*Formenreihe*], presented as step-by-step routines emerging through a rule of operation on a basis, are central to his characterization of logic, above all his treatment of generality.⁷ Such series themselves constitute bases (themselves, presumably, well-founded) to which are applied operational (recursively generated) rules, first and foremost using the generalized version of the Sheffer stroke of

⁵ Floyd forthcoming c. Recently discovered notebooks of Turing’s on notation indicate his interest in 1939-1942 in the history of philosophers’ quests for a universal language; cf. Hatton and Hodges 2015.

⁶ TLP 5.5563, Kremer 1992; Weiss 2013.

⁷ Floyd 2001a, Rogers and Wehmeier 2012, Ricketts 2013, Weiss 2013.

joint denial. The totality of these rules provide the operational or “calculative” filaments that, as a whole, allow us to clarify and make manifest internal logical relations among things that are there to be said. The general characterization of this, conceived as a general characterization of the notion of a truth operation applied to a propositional basis, is given in the “general propositional form” (TLP 6).

In his intermediate period of development (c. 1929-1933), simplicity is re-conceived by Wittgenstein. As is well-known, he had rejected by then the idea that at the most fundamental level there must be a mutual independence among all elementary propositions. Framing a different view of the bases of logical operations, he formulated the notion of a *Satzsystem*, a grammatical system of propositions, and insisted that there is a variety of differing *Satzsysteme*. This variety he took to be reflected in alternative analyses of given fields of experience. He therefore took simplicity – and indeed the very idea of a proposition, possibly true or possibly false – to be something relative to a *Satzsystem*. Thus now there are different *kinds* of simplicity. For example, the color field might be analyzed in terms of the primary colors, or in terms of the color octahedron. No longer was the truth table taken to be a fundamental mode of presentation of all propositions. Instead, logic was taken to be more complex, no longer unified by any general propositional form.

This “middle period” view, with respect to simplicity, represents a kind of hybrid compromise with the *Tractatus*. Simples are the endpoints of analysis, and constitute a basis for a given *Satzsystem*. Each system is conceived to be a “calculus” of its own. Analysis remains, as in the *Tractatus*, well-founded, bottoming out in such a way that sharp answers, Yes or No, are to be delivered to the totality of questions that might be posed about what may or may not be said relative to this or that sayable within the *Satzsystem*. Simplicity is system-relative, a feature of rule- or grammar-clusters. Undefinability (“impossibility of analysis”) is taken to evince a feature of a form of representation that we ourselves have set down, rather than something that will (ideally) show forth anyway, regardless of what we do, as was the case in the *Tractatus* (cf. BT p.

201).⁸ Simplicity is thus “subjective” in being something reflected in our choices of *Satzsysteme*; it is “objective” within an analysis of the grammar of a particular *Satzsystem*, being well-founded. *Satzsysteme* constitute – to vary a metaphor of Michael Dummett’s that he devised for what he took to be the later Wittgenstein’s philosophy of mathematics – a series of separate islands or calculi sitting in a fragmented archipelago.⁹

Nevertheless, the idea of a formal series, produced by an articulated operation or rule applied in a step-by-step manner from a given basis, remained a focus of Wittgenstein’s conception of logic during this time. No longer construed in terms of the “general propositional form”, this notion was reconceived as a formal aspect or “feature” [*Zug*] of certain internally related steps, for example, those emerging in proofs by mathematical induction on the number series. Number was still characterized, in early discussions of the 1930s, by Wittgenstein’s operational concept of form: $[0, \xi, \xi + 1]$ ¹⁰ However, a worry about how we get our “hold” on the natural numbers – i.e., how we make sense of the *first* step in this series, and our remaining “in” it – was an important driving force in Wittgenstein’s Cambridge lectures and writings of the 1930s; he did not want a “nebulous” introduction of the numbers.¹¹ Characteristically, he still regarded such a formal operation as one that would be well-founded. He struggled in his Cambridge lectures of the 1930s with how satisfactorily to formulate the very idea of definition by recursion, isolating the need for a uniqueness rule in an equational specification by May 1932.¹²

By the time of *The Blue Book* (fall of 1933) Wittgenstein had made real progress in chasing down a different kind of answer to the question of “first” steps and formal totalities or systems.

⁸ On the idea of “showing forth anyway” in the TLP see Narboux 2014.

⁹ Dummett 1959, 326.

¹⁰ PR 109, 158, PG 431-433, BT 469.

¹¹ On this see WCLM 6b: 17 (May 1932), discussed in Marion and Okada forthcoming, Goldfarb forthcoming.

¹² Cf. Goodstein 1945, p. 407n. As von Plato (2014), Marion and Okada (forthcoming) explain, this was a genuine contribution to logic, and one that I suspect led Wittgenstein to teach “Philosophy for Mathematicians” in the fall. It is quite possible that Turing attended this course in either 1932 or 1932-3, on which see Floyd forthcoming c.

Simplicity at this stage is relativized to language-games. And at this stage of his thinking the amalgamation of separate systems – hence simplicity itself – is represented in a serially presented, quasi-evolutionary, step-by-step manner. Language-games are themselves taken to be “simple”, but also dynamic “objects of comparison”, embeddable in part within one another through projection, addition, and reconfiguration. Here and in *The Brown Book* Wittgenstein rejects the whole idea of “undefinability” as unclear, so that his middle period obsession with internal, well-founded totalities of grammar is jettisoned. Moreover, by the time of *The Brown Book* Wittgenstein broke through to what is a crucial innovation, his explicitly anthropological stance on language-games, shown in his discussion of tribes.¹³

But on my view there was still another, crucial, conceptual step to be taken. And it is this that is reflected in the change of style evinced in the PI. It was only in 1936-7 that Wittgenstein was able to embrace the most radical idea, the idea that simplicity is, not simply relative to a system, or a tribe, or even a culture, but something fluid. Necessarily and absolutely fluid. Necessarily and absolutely a matter of ongoing discussion.

Wittgenstein has thus returned to themes proper to the *Tractatus*, in an odd sort of arc. For the very sake of exactness, to acknowledge the nature of the logical, he shows that we always have to start somewhere, we always take something to be simple, evident, unproblematic. There are always “simples”, in the sense that we always take a step from somewhere in reasoning or rule-following. Yet there is drift in our lives and discussions, and step-by-step procedures are sometimes partially defined, dropped, detached, refashioned, amalgamated, re-ascribed, re-oriented, re-framed, and revisited. What is a “simple step” may be contested, shown not to be so. Analysis goes on, but against a more dynamic backdrop. It has no general form, yet it moves from within a unified philosophical – and logical – perspective.

What is different here, in relation to the intermediate, relativistic position, is that there remains something absolute and robust about

¹³ On this see Engelmann 2013a,b; cf. Sen 2003 also on the influence of Gramsci and Sraffa.

the logical, conceived in terms of step-by-step symbolic procedures, something that remains a unity, that prevents the idea of a *Satzsysteme* from hardening into a static, divided archipelago of conventional schemes, or an artificially ordered series of such. My suggestion is that Turing's analysis of a logical "step" in his "On computable numbers" allowed Wittgenstein to have faith in this as a robust way to conceive the nature of the logical.

To be sure, Turing's paper establishes that there is not one system or procedure within which we can decide questions Yes or No as to what follows from a set of axioms: this is his (negative) resolution of Hilbert's *Entscheidungsproblem*. Instead, there are many interwoven, recognizable step-by-step procedures. But these are, so to speak, the "overlapping" fibers that hold the realm of the logical together as a totality, conceptually.

And thus it is perfectly logical for Wittgenstein to have surrendered the ideal of a "natural", "well-ordered", or "gap-free" presentation of *his* thoughts in PI. An album of partial snapshots suits his view better. The "gap-free" ideal is precisely the one alluded to explicitly in PI as a form of presentation of his ideas Wittgenstein realized he had to give up (PI Foreword, 67).¹⁴ Neither Turing's nor Wittgenstein's conception offers a gap-free [*luckenlose*] presentation of logic in *general*. The command structures in terms of which Turing presented his "machines" are reminiscent of the command tables of *The Brown Book*. From one point of view they are static, mere quintuples of signs. Yet each represents, from an end-user point of view, a process, something dynamic (like Wittgenstein's presentations of language-games). What Turing *did*, by way of this dynamism, was to give us a penetrating analysis of precisely that gap-free aspect of the logical that Frege and Russell explicitly sought for in proofs *in* logic. He rigorously manifests that notion's *internal* limits.

¹⁴ I differ with the elimination of Frege's term "gap free" [*luckenlose*] from Anscombe's and Hacker and Schulte's English translations. This takes the PI out of the orbit of the logical. Wittgenstein, it seems to me, is clearly making direct allusions to the tradition here. Cf. Frege 1879/1967 Preface IV, VI, 1884/1974 §90. Compare Wittgenstein's 1938 drafted Foreword, where he wrote that he had supposed that the thoughts should be well-ordered in a step-by-step series [*dass die Gedanken darin von einem Gegenstand zum andern in wohlgeordnete[n] <<einer>> Reihe fortschritten sollten*] (117, 121-2; cf. his revisions in 2003 to FF, and the *Zwischenfassung* (hereafter "ZF") Forewords).

What Turing offered through his *informal* comparison between a human calculator and a machine *domesticates* the gap-free aspect of logic, just that aspect Wittgenstein had previously presented as central to logic with his earlier notion of a form series. In Turing's paper, "gap-free" reasoning, presented by formulating operational rules for generating internally related steps with a finite symbolism indefinitely far, is presented in terms of his "machines". Turing shows through his analogy how we can "break off" from a (logical) routine and begin another, coming to rest with our philosophical questions – at least momentarily – by proceeding, and then, later on, refashion what is "simple" by beginning again (cf. PI 133).

After 1936-7 Wittgenstein's accent is on the fact that each step we take to be simple may itself be unwound as complex: moved, critically reflected upon, contested, dropped. And then that contestation or amalgamation or dropping may itself, with its simples, in turn be scrutinized anew. This is first-order philosophy, logic done without falling back upon a hierarchy of types or metalanguages, except for particular purposes, which may be pursued and moved on from (cf. PI 121).

Of course, I am not offering here a monolithic account of Wittgenstein's development, but only a suggestion about one proximal stimulus and factor, one I take to have been quite significant and which has not been broached in the literature. For example, Wittgenstein had long rejected the idea of a "metalevel" from which to solve his philosophical problems.¹⁵ But now, in 1936-7, I believe that his perspective has finally gelled. From Russell's (and Gödel's) vertical, top-to-bottom and bottom-to-top axiomatic picture of the hierarchy, Wittgenstein has moved away, conceptually: through a disordered archipelago of well-founded but conventional calculi, to a serial ordering of language-games, and, ultimately, to a fluid, modular setting for human procedures, embedded in our ordinary ways of talking as these have evolved in everyday life. Turing's analysis of logic forwards just such a view. Our daily lives with "apps" today evince and instantiate the perspective. They are *Lebensformen*.

¹⁵ PR §153, WVC 121,123,136, WCLM 3:29 (May 1931), 6:4 (May 1933), BT §109.

On Wittgenstein's mature view of simplicity, the simples (say, those inherent in a particular set of commands) are firm points to grab onto, just as water provides a firm basis for swimming, if we move our limbs in the right ways. But logic is dynamic, embedded in our procedures and our discussions and arguments. We analyze and we follow rules, oftentimes only partway, taking in certain aspects (articulated within analyses) as beginning points for procedures, pursuing these until the rules (concepts, procedures) give out on us. In water, we may sink if we do not swim, remaining active and passive in the right ways. Similarly, in fashioning and scrutinizing and offloading our step-by-step procedures in language, we require harmonies and agreements among us (*Übereinstimmungen*), and these too require our active interventions and passivities along the way: our discussions, our agreements and disagreements with one another, the give and take of our talk, our interests, and our actions, our appreciation of the *point* of a step-by-step routine (cf. PI 242).

Given the fluidity of simplicity, it is not at all clear, *a priori*, what analysis will look like, where it will begin and where it will end: there is no logical essence in any particular place. Yet this does not imply that there are no necessities or analyses, that simplicity is merely subjective, or relative to a system-choice, much less that we cannot make sense of the general notion of following a rule, or the idea of mathematical certainty, or of analysis. Instead, we have to turn our reflections on logic around, toward the direction of our real needs and requirements (PI 108). Algorithms are neutral in some ways, and hardly neutral in others. In *general*, they swim in an intensional soup.

Turing's analysis of the idea of a logical "step" in a formal system shows that step-by-step symbolic procedures do evince necessities, but these do not and cannot depend upon one overarching proceduralization of thought. Instead, at the heart of his analysis lies the schematic idea of proceduralizing, of amalgamating routines, as such. This is a perfectly rigorous, robust analysis, as I shall explain further in Section 2. It is not grounded in any particular place, and is therefore absolute. Its ubiquity in life is held up in the settings of human conversation and the need to

offload routines to enhance what we say and do, not only as individuals, but together, communally.

It is surely significant that it was not until 1936-7 that Wittgenstein surrendered his mode of proceeding with language-games step-by-step, in a linear manner, as he had done in *The Blue Book* and *The Brown Book*. Is it not just, as he later complained, that *The Brown Book*'s form is "boring and artificial", though that may be true.¹⁶ The turn reflects, not *merely* a matter of style, but, it seems to me, of logic and method. He was driven to the later adaptation, evinced in the way PI is written, because, with Turing's analysis in hand, he now realized that he could – or should – continually detach, move, rearrange, amalgamate and reconfigure motifs and pieces of procedure and thought and conversation (and its ending) within one another without end, erasing and revising starting parts of thoughts once written down, shifting their force, revisiting themes and drawing out variations in a multitude of dimensions at differing scales, endlessly.

The serial, evolutionary linearity of *The Brown Book*'s presentation was insufficiently clear. His new style expresses the fact that he now had faith in the unity of logic again, having surrendered the earlier, relativized ideal of simplicity. Now, at last, the games have depth, even if not along one scale. The rearrangements, the cobblings, hold up – within our step-by-step proceedings, which may be endlessly interwoven with one another, dropped, amalgamated, seen to be fruitless or empty, and so on. The logic in language is like an ancient, living city, not like an evolved, historical, serially-presentable series of features of culture as such (PI 18).

The literary result is an "album", a landscape of shuffling scenarios of voices and variations, echoing and cancelling one another with modulations, self-confessions, re-phrasings, and artificial and natural snapshots of philosophical activity.¹⁷ This is

¹⁶ WC #204, 257 (November 20, 1936). Engelmann 2013b (chapter 4.4) quotes and analyzes this letter and MS 142. His view is that Wittgenstein "does not seem to believe that there were huge changes in the new version of his work" (p. 207). But he notes some important differences, especially the suppression of tribes and the use of the "genetic method" of laying Wittgenstein's *Tractatus* views down against his mature ones.

¹⁷ Cf. Pichler 2004.

dialecticity with precision and without loss, rigorous in its own way, a specific form of schematicity without logical schemata. It is an analysis of analysis as such, and therefore free of any *particular* choice of formal, sequenced venue. Spades are turned, and turnings are discussed. There are simples, but they are fluid, elicited from the reader at a variety of different points, from different perspectives, for different reasons.

Wittgenstein's later style has been called by Goldfarb a cultivation of "intentional naïveté".¹⁸ We might think of this as a way of figuring and eliciting from the reader a view of what it is to think through to the simple, the unvarnished and the natural, an arena which is, at the same time, one of unending depth and sophistication, a series of arguments about what is to count as simple, straightforward, obvious, or given. Wittgenstein had thoroughly reflected on the very notion of logic, logical analysis, philosophy, itself. That reflection gets him back, ultimately, to *Lebensformen*.¹⁹

The structure of the remainder of this essay will trace out the arc of this drive to reconceive the notion of simplicity. Section 2 analyzes the initial occurrences of *Lebensform* in Wittgenstein's writings of 1936-7, showing how Wittgenstein came to re-focus on simplicity at this time, just as he pressed toward a more general application of his ideas, framed his mature style of writing, and drafted the earliest versions of PI. Section 3 offers an account of Turing's 1936/7 paper, explaining its Wittgensteinian air, and

¹⁸ Goldfarb 1983, 269.

¹⁹ A later use of *Lebensformen* (probably after 1946), published in RPP I §630 makes this association with simplicity clear:

1298.«5» Statt des Unzerlegbaren, Spezifischen, Undefinierbaren: die Tatsache, daß wir so und so handeln, z.B., gewisse Handlungen strafen, den Tatbestand so und so feststellen, Befehle geben, Berichte erstatten, Farben beschreiben, uns fuer die Gefuehle der Andern interessieren. Das hinzunehmende, gegebene – könnte man sagen – seien Tatsachen des Lebens. //seien Lebensformen.//

[Instead of the unanalyzable, the specific, the undefinable: the fact that we act in such-and-such a way, for example, that we punish certain actions, determine the facts as such-and-such, give commands, make reports, describe colors, interest ourselves in the feelings of others.

That which is accepted, given – one might say – are facts of life. //are *Lebensformen*.//]

According to von Waedelstaedt, the origins of this are in MS 133, 284 (7.11.1946), TS 229, 333 (1947) and MS 144, 102 (1949). I depart here from Hacker and Schulte's translation of PI 345, which, like Anscombe, puts necessity into the substantive *hinzunehmende* with a "must" that I do not take to be there.

insisting, against usual views, that it is not so much philosophy of mind or mathematics that are the areas of interest here, but instead the very nature of logic. Section 4 erects a conceptual frame for each of the five occurrences of *Lebensform* in PI, stressing the importance of his surrendering his reliance on the notion of *culture* (still at work in *The Brown Book*) and drawing in a discussion of his mature conception of what logic *is*. I will work through a series of analogical notions that I take to be central to Wittgenstein's mature conception of analysis and logical necessity: the notions of aspect, chain, ordering, and scaffolding, with their attendant notions of binding, weaving, holding together in step-by-step surveys of reasoning. I show how these ideas were endemic in the history of logic, and indicate how Wittgenstein is reworking them by use of his notion of *Lebensform*.

The occurrences of *Lebensform* proceed in a regressive manner, heading towards what is, in the end the “given”, i.e., the ultimate starting point for what Wittgenstein hoped would be a future synthesis of his philosophical thought, though this was of course never completed. The scheme for Section 4 is this:

4. *Lebensform* as a Logical Notion
 - 4.1 The Idea of a “Chain” of Reasoning
 - 4.2 PI 19: Form of Life and Logical Aspects, not Culture and not Grammar
 - 4.3 PI 23-25: Natural History and Logic, Regressive Analysis, Concatenation
 - 4.4 PI 240-242: Agreement in Judgments, Scaffolding
 - 4.5 PPF i 1-2: *Übersichtlichkeit* in the Tapestry of Life (*Lebensteppich*)
 - 4.6 PPF xi 343-363: The Ribbon of Life (*Band des Lebens*), *Lebensform* as the “Given”

2. Early Versions of the PI: The Emergence of Wittgenstein's Later Style

The first occurrence of *Lebensform* is in MS 115, 239, from 1936, Wittgenstein's attempted translation and revision of *The Brown Book* (posthumously published as “Eine Philosophische Betrachtung”, hereafter “EPB”). This was undertaken in earnest in the fall of 1936, in Norway, and abandoned by November. His use of *Lebensform* occurs in the midst of a struggle over imagining a “use

of language” or “language”, “and that means again a *Lebensform/Form des Lebens*” (there are several variants here) in which one fixes a “gap” [*Kluft*] between dark blue or red and light blue or red.²⁰ Wittgenstein had previously added to his manuscript, for the first time, a reference to Plato’s *Theaetetus* that connects the dialogue directly to the idea of ultimate simples in the *Tractatus* (EPB 121).

By EPB 292 he gave up, writing that “This whole ‘attempted revision’ from page 118 to here [the attempted revision of *The Brown Book*] is worthless”. Immediately thereafter, he faced a tremendous intellectual and personal crisis.

However, that same dark month, alone in Norway, filled with confessions and self-doubts, he began again anew, and drew in the notion of *Lebensform*. He drafted what would become, over the course of the rest of the winter and the spring of 1937, the *Urfassung* (origin-manuscript) of the PI (MS 142, hereafter “UF”).²¹ The first 76 pages or so (through the first part of UF 86) were composed before Christmas in 1936. The rest was written in the spring of 1937, before May 1, when he left Norway for Cambridge. The whole amounts to drafts of the material in PI 1-189(a). It contains the origin of PI 18-23, with “*Lebensform*” occurring in the source of PI 19 (the imagined language of only orders in battle, and countless other things, UF 16) and that of PI 23 (“‘language-game’ is meant to emphasize the fact that the speaking of language is part of a form of activity” (*Form der Tätigkeit*, “or”, as Wittgenstein says, of “a *Lebensform*”)).²² It ends with his claim that the interlocutor’s question about the steps *not* being “determined” by the algebraical formula “contains a mistake” (cf. PI 189).

When he returned to Cambridge in summer 1937 he had a typescript made of the whole (TS 220). He showed it to G.E.

²⁰ MS 115, 239, EPB 108a:

10[7|8] Umgekehrt könnte ich mir auch einen Sprachgebrauch «eine Sprache» (& das heißt wieder eine Lebensform «Form des Lebens») denken, der «die» zwischen Dunkelblau«rot» & Hellblau«rot» eine Kluft befestigt.

10[7|8] Conversely, I could also think of a language usage “a language” (that is, a life form “form of life”), which “fixes” a gap between dark blue “red” and “light blue”.

²¹ For a critical edition of these versions see Wittgenstein 2001.

²² “*Lebensform*” is, however, only a variant at UF §24, whereas it makes its way clearly in as a substitute for “activity” in the *Frühfassung* 24 (hereafter “FF”) in 1938.

Moore and some other colleagues, discussing it with (among others) Turing. Moore would later say to Rhees that Wittgenstein told him at this time that *The Brown Book* had followed the wrong or “false” method [*falsche Methoden*], but now he had come to apply the “right” or “correct” method [*die richtigen Methode*]. Moore said he did not know what Wittgenstein meant.²³ But we can guess, with the power of hindsight.

A crucial earmark of the UF’s importance is that it is here, for the first time, that Wittgenstein thematizes simplicity in the manner of his mature work. He added in the fall for the first time the swath of remarks beginning with the quotation from Socrates’s dream in the *Theaetetus*: here Wittgenstein models a language-game with arrangements of colored squares (UF 47(49); cf. PI 46-76). He directly follows this swath with his remarks about “family resemblances” (UF 64ff.; PI 67ff.). The swath written through Christmas 1936 ends with the question, “In what way is logic sublime?” (UF 86, cf. PI 89). And Wittgenstein went on writing through the spring of 1937 implementing what Engelmann has called the “genetic” method: the dialectical juxtaposition of his mature thinking with his *Tractatus* thinking, to signal effect.²⁴

Notebook 152, written sometime in the spring of 1937, engages explicitly with the problems of simplicity and generality raised by the *Theaetetus* remarks. We should regard it as delving into the question with which Wittgenstein had broken off his manuscript at the end of 1936: In what sense are logic (and simplicity) something sublime? It begins by writing out several examples drawn from the theory of continued fractions. In these examples, the infinite descent characteristic of an irrational number is controlled through systematic substitutions of infinitely long terms into finite terms recognizable in the context of definite equations. Calculation proceeds, so to speak, analogically, by substituting like-looking terms for indefinitely long procedures (infinitely descending series) into variables that can be manipulated by means of algebra.

²³ Rhees wrote (EPB, 12-13) that the manuscript included TS 221 and that Moore gave this manuscript to him to read in the spring of 1938. But the editors of Wittgenstein 2009 disagree (xviii f.), saying that Wittgenstein only had TS 220 in Cambridge with him in the summer of 1937.

²⁴ See Engelmann 2013a for a discussion of this new method.

Wittgenstein immediately moves from this warmup to consider the difficulties about simplicity raised by such examples and his new view. He refers back to “the problem” of UF 47, where the characterization of the language-game (of UF 38) is transformed into a series of three sentence-like commands, ordered n -tuples of the three letters “a”, “b” and “c” directing the drawing of oriented lines (as in a Turing machine). He asks what would happen if player B, programmed with the three commands, “ran out of letters”, and points out that there are a variety of circumstances in which we would say differing things (e.g., that B had gone out of the game; that B can no longer “run”). He is pressing hard on the notion of the fluidity of the simples, moving from an example illustrating the scope and nature of well-founded procedures – infinite descent in the context of step-by-step procedures with irrational numbers – to the issue of how the list of finite command strings serves to illuminate the issue of simples in the context of analysis into step-by-step procedures generally.²⁵

Notebook 152 continues, but on p. 39 Wittgenstein reprises, recommencing with a variety of ways of reacting to the Augustine quotation. At this point he switches his orientation. He now relies on references to MS 142 (the first half of UF), having decided to rely on *it* (rather than EPB) to move forward. By the end of the notebook he returns again to the ideas of simplicity and the sublimity of logic. He closes the notebook this way:

The simple as a sublime term and the simple as an important form of the representation but with domestic application (152, 96).

Now why would Turing’s 1936/7 paper be relevant to *this*? The answer is that in his paper (which, as we have said, Wittgenstein may well have had on hand in the spring of 1937) Turing had successfully *analyzed* the very notion of a “step” in a formal system, the very idea of “calculation in a logic”. He did this by reducing the idea of logic as a calculus to intuitive, picturesque, simple-minded terms. He boiled down the numerous existing formal systems and

²⁵ Incidentally, in Turing 1936/7 the “computable” real numbers are given by “circle-free” machines, the ones that will run, approximating a real number forever, computing *via* the unlimited tape an infinite series of 0’s and 1’s. They never run out of ink, though they begin with a finite stock of indefinitely reusable symbols, so these never run out.

proof-theoretic approaches of his time by comparing their uses to the most human, simple, elements of ordinary calculation: a set of shareable commands, a piece of paper, a pencil, a bit of memory, routines and step-by-step movements. Dare I say it: a form of life.

We could conceive this, as Wittgenstein did, as an embedding of possible step-by-step procedures, circumscribed from within, in life. Turing used his standardized, informal, analogical characterization of a computable procedure to prove that there can be no *general* decision procedure to determine from *within* the boundaries of the logical whether or not a sentence is a logical one. That is, he resolved (negatively) the *Entscheidungsproblem*. I think this fascinated Wittgenstein. If I am anywhere near right, the success of this analysis sparked or at least gave Wittgenstein confidence in his mature philosophical method of writing.

First, a few more historical facts. Turing had finished his paper by April 1936, and submitted it in May, continuing work on it during that summer in Cambridge (see note 3). In August 1936 Wittgenstein left for Norway. He may have seen a draft of Turing's paper over the summer, before he left, or discussed it with one of their mutual friends; even if he did not, he would probably have heard a buzz about it in Cambridge circles, and been made aware of the (negative) result. As we have said, the fruit of the fall of 1936 in Norway for Wittgenstein was his abandonment of *The Brown Book* and his drafting of, first, the origins of PI 1-80 (before Christmas), and then, in the spring, the rest of UF, encompassing the first 189 sections of the PI.

Turing sent Wittgenstein an offprint of "On computable numbers" sometime before February 11, 1937.²⁶ By that summer, back in Cambridge and armed with UF, Wittgenstein not only met with Moore. He also discussed philosophy with Turing, who was back at Cambridge for a visit between his years at Princeton. They met with Alister Watson, a close student of Wittgenstein's and

²⁶ See note 3.

fellow Kingsman with Turing who was working on a paper on the foundations of mathematics.²⁷

These discussions pursued issues in the foundations of mathematics stretching back to the Greeks, and included exchanges about Gödel's and other limitative results, including Church's and Turing's. Turing was interested enough in pursuing these issues that he considered writing a sequel to his 1936/7 paper, in which he would develop the foundations of computable analysis.²⁸ When he returned to Cambridge in 1939, and was teaching the mathematical logic course, he sat in on Wittgenstein's lectures on the foundations of mathematics (LFM). In future writings Turing would evince and explicitly acknowledge his debts to these discussions, as I have shown elsewhere.²⁹

What is important for our present discussion is that like Wittgenstein, Turing would drive more and more explicitly toward the foundational importance of ordinary phraseology, human cultural searches, and the importance of "common sense".³⁰ The chronology mirrored the regressive method for him as well. Wittgenstein himself stressed this feature in later writings, remarking (in 1947) that "Turing's 'Machines': these are *humans* who calculate" – just before he reconstructed Turing's argument in "On computable numbers" in terms of games.³¹

A significant historical indicator of the impact of these discussions on Wittgenstein is that immediately after he finished his summer discussions with Turing and Watson, Wittgenstein returned to Norway and composed, over that autumn and subsequent spring of 1938, the first full length draft of the PI (the

²⁷ Hodges 1983/2012. The paper was Watson 1938, in which he thanked both Wittgenstein and Turing for his knowledge of Gödel's theorems. Wittgenstein admired the paper, on which see Floyd 2001b, Floyd and Putnam 2012, Floyd forthcoming c.

²⁸ Hodges tells us that Turing was "overoptimistic" about rewriting the foundations of analysis at this time (1999, 19). Gandy 1988, 82, n. 25 explains that it was the fact that not all real numbers have unique representations as binary decimals that led Turing away from the topic (cf. Turing 1937, where at the suggestion of Bernays Turing includes a reference to Brouwer's overlapping sequences treatment of real numbers). Serving as examiner of Goodstein's PhD thesis in 1938, Turing noted large errors due to these ambiguities of representation.

²⁹ Floyd 2013a, forthcoming c.

³⁰ Floyd forthcoming c.

³¹ RPP 1096-7 (from MS 135, p. 118). For discussion see Floyd 2012c.

so-called *Frühfassung*, or “Early Version”, hereafter “FF”).³² In his initial manuscript remarks, begun in August 1937 in Cambridge and continued after his arrival in Norway in September, he worked up the first draft of his famous remarks about the idea of a machine symbolizing its own mode of action³³, as well as his first series of remarks probing the philosophical relevance of Gödel’s incompleteness theorems.³⁴ It is reasonable to take these remarks to register the stimulus of the 1937 conversations with Turing and Watson in Cambridge.

FF is a draft of PI that extends UF past the origin of PI 189 by applying the perspective to the foundations of logic and mathematics. Wittgenstein would eventually submit this to Cambridge University Press as part of his bid to become Professor. In this manuscript his mature view of simplicity is applied, and a different focus from the one worked up in the years 1929-1934 emerges. Wittgenstein always envisioned his second book, until late in his life, as having a particular structure, one which FF instantiated (cf. PPF xiv 372). In the first part, the nature of sense, thought, and meaning would be clarified. In the second part this conception would be applied to logic, philosophy, intention, idealism, and, finally, the foundations of mathematics. That last part was a constraint on success, for Wittgenstein, as it has not always been for his readers. He bound himself to it, as a standard of rigor. (The *Big Typescript* (TS 213, hereafter “BT”) adheres to this form.)

If I am right, it is the new ideal of simplicity that shapes this material. This explains Wittgenstein’s odd yet focussed determination to make sense of *first* steps into logic and mathematics. These are now subjected to the idea that we cannot really experience or ideate them directly, in an empiricist’s way, or even in terms of the metaphor of a “step”, much less a theory of logic or concepts. The elementary primitiveness of the examples

³² This is a continuation of TS 220 into what would become TS 221 (with a Foreword in TS 225). Floyd 1991 analyzes this context line by line, with attention to the manuscript composition.

³³ 119, 28; cf. PI 193, FF 349, RFM I 122. As for precursor remarks, at WCLM 5: 43 (May 1930) Wittgenstein does discuss the idea of looking at a machine as an expression of a rule or possible motion. But not the machine symbolizing its *own* actions.

³⁴ MS 118, 211ff., cf. FF 383ff; cf. RFM I Appendix III.

stalked in this manuscript is then not a merely pedagogical conceit of Wittgenstein's, much less a reflection of his ignorance of mathematics or a kind of ideological commitment to the non-technical.³⁵ Instead, it is a logical must, demanded by – but also clarified in a reaction to – Turing's work. Wittgenstein had to regress backward, to the simples, and see them rightly.

The remarks in the second part of FF were never satisfactorily finished, and Wittgenstein dropped them from subsequent versions of the PI. The issues were certainly at stake in the exchanges that continued between Turing and Wittgenstein in 1939 (LFM).³⁶ It seems Wittgenstein may have hoped eventually to return to the material, and keep the original mode of organization envisioned for his book, but he never completed this task. Instead, the remarks extending the UF were published posthumously, as Part I of *Remarks on the Foundations of Mathematics* (hereafter “RFM”). It is not my aim here to give a reading of them along the lines I have just suggested, though elsewhere I hope to do so. I have only noted the sequence of historical steps to prepare my logical reconstruction of a larger scheme.

3. Turing's “On computable numbers, with an application to the Entscheidungsproblem” (1936/7)

A brief remark about how I read Turing is incumbent at this point. For it is customary – and not only among Wittgenstein scholars – to regard Turing as a kind of arch mechanist in the philosophy of mind, or at least, a kind of computationalist-behaviorist-reductionist. Read in this way, Turing is a philosophical enemy of Wittgenstein's, if not a clear object of philosophical attack in PI.³⁷ The tradition of reading Turing psychologically, in this way, began quite early, invited, to some extent, by Turing himself. First, he was embraced as a kind of hero by functionalists, AI-enthusiasts, and cognitive scientists. And then he was criticized as an arch mechanist by philosophers, including Gödel, who argued

³⁵ This is also emphasized in Floyd and Mühlhölzer (manuscript), in relation to Wittgenstein's 1941-44 criticisms of the “extensional” point of view on the real numbers.

³⁶ Floyd forthcoming c.

³⁷ Kripke 1981, Shanker 1998, the latter discussed in Floyd 2012c.

vociferously against what he regarded as a dogma in Turing's work: the assumption that states of mind of humans are finite in character, and discrete.³⁸ The opposing legacy of computationalism and functionalism in the philosophy of mind is well-known.

There is insufficient space here to argue against this widely-held view of Turing, though I have provided varied grounds for an alternative interpretation elsewhere.³⁹ Here I shall just sketch another, more Wittgensteinian reading of Turing. It will be enough to insist that although the notion of a "state of mind" does appear in Turing's paper, it does not do any substantial metaphysical or argumentative work there. After all, mathematical theorems cannot be proved on the basis of theories of mental states. Instead, Turing's remarks form part of an *analogy*, a powerful one that Wittgenstein appreciated. The important point is that this analogy was just perfect for the Hilbertian context into which Turing was inserting his work.

First let us be clear about the *logic* of the situation in 1936. Turing's analysis of a "step" in a formal system – as I have said, partly inspired by Wittgenstein's lectures and dictations – is a logical *must*. To show that there is no *general* procedure or "definite method" for determining, in a finite number of steps, whether or not a sentence of a formal system does or does not follow from the axioms, one must first clarify what in general it *would* have been to have such a general method in hand. (In contrast, a positive proof that there *is* a definite method – or formal proof, or algorithm – merely requires that we produce it.) In other words, in order to show what formalisms *cannot* do – which is above all what Turing showed in "On computable numbers" – one is obliged to give a general analysis of the very idea of a "formal system", or a "systematic procedure", *in the relevant Hilbertian sense*. One must clarify what the nature of formal systems are in *general*, the very idea of one.

³⁸ See the note Gödel added to his "Some remarks on the undecidability results" 1972a, in Gödel 1990, 304. Gödel accuses Turing of a "philosophical error" in failing to admit that "mind, in its use, is not static, but constantly developing", as if the appropriateness of Turing's analysis turns on denying that mental states might form a continuous series. Critical discussions of this in relation to Turing are given in Webb 1990 and Floyd 2012c.

³⁹ Floyd 2012c, 2013a, forthcoming c.

To do this one cannot simply write down another formalism, or keep on urging people to do so, as did Carnap.⁴⁰ That will not do. Nor may one invoke the idea of a “metalevel”. For that would have no hope of analyzing the *general* idea of a “step” in a formal system. In the end one cannot resort directly to formalized means – although whatever one writes down, it must be provably robust enough to handle any and all possible formalisms of the relevant kind *and* be able itself to serve as one. The very idea of writing down a “letter”, “phrase” or “symbol” in the context of a step-by-step procedure must be indefinitely schematized. And this is what Turing did.

The logical need for something vivid and “profoundly ordinary”⁴¹ is what Turing’s analysis supplied, and that is why it is a remarkable piece of “applied philosophy”⁴², a “paradigm of philosophical analysis”.⁴³ He *unvarnished* symbolic logic, making the idea of a formal system (in the relevant sense) *plain*. Turing analyses what a formal system *is* by showing us what a formal system is *for*, how it is *used*. The analysis explicitly turns on an *informal* “comparison” between a human calculator using pencil and paper, following a series of determinate commands, and a mechanical process implemented by a machine.⁴⁴ It is a “language-game”, an object of comparison (cf. PI 130-131), drawing the human, with its “form of life”, vividly into the foundations. Probably inspired by reading *The Blue Book* and/or *The Brown Book*, Turing realized how to characterize the relevant notion in a general way.

As Gandy wrote (1988, 85), “Let us praise the uncluttered mind”. Turing’s lack of a theory of mind, or commitment to any particular theory of mentality, is confirmed by the argumentation in his (1936/7). In the course of the paper, Turing explicitly *replaces* the notion of a “state of mind” with a shareable counterpart: a command that may be dropped and passed off to another

⁴⁰ This, I think, is why Wittgenstein said he would throw up his milkshake at the bar if Findlay went on talking about Carnap’s merits (Findlay 1985, 58). On Turing vs. Carnap see Floyd 2012b.

⁴¹ Hodges 1983/2012, 96.

⁴² Davis 1982, 14.

⁴³ Gandy 1988, 6; cf. Floyd forthcoming c.

⁴⁴ Turing 1936/7 §1.

computer.⁴⁵ And in terms of *this* object of comparison he is able to (negatively) resolve the *Entscheidungsproblem* for logic. Ramsey had once called this a “leading problem of mathematical logic”, and solved a partial form of it (positively).⁴⁶ Turing showed that there is nothing like this partial result for the general case, for the whole of logic (in the relevant sense). And so, just as Wittgenstein remarked, any further “leading problem of mathematical logic” could be regarded as a mathematical problem “like any other”, i.e., one that will not dislodge or re-ground our general conception of logic (cf. PI 124).

Another indication that a Wittgensteinian air permeates Turing’s use of this analysis is a striking feature of the particular argument he gives. His argument is, as has rarely been noticed – though it was explicitly stressed by Wittgenstein⁴⁷ – *idiosyncratic*. For Turing’s argument does *not* rely on negation. This means that it applies, not only to classical logic, but any other formal system of logic, be it intuitionistic, constructivist, finitist, and so on. It is not *really* a diagonal argument of the kind used, for example, in Gödel’s incompleteness argument (1931).

What Turing does instead is to show that if there *were* a general decision procedure for determining whether or not a sentence in a formal system does or does not follow from the axioms, this would give us the wherewithal to construct a defective machine, one that is infected with an *empty* command, something of the form, “Do what you do”. I call it the “Do-what-you-do” machine.⁴⁸

Imagine drawing a card in a game that says “Do what you do”. This command is something even more bare than a tautology. We can *see*, by inspection, that although it is definable, it *cannot* be followed. The difficulty takes us beyond what step-by-step procedures *are* – unsupplemented by a specific context. Of course, in everyday life the form of words “do what you do” might find a use (if I am, for example, coaching you to continue onward with a

⁴⁵ Turing 1936/7, 253 (III of §9).

⁴⁶ Ramsey’s remark is in his 1930, 26; cf. PI 124d.

⁴⁷ RPP I 1096ff, discussed in Floyd 2012c.

⁴⁸ The argument is reconstructed in detail, with reference to Wittgenstein’s remarks and Turing 1936/7, in Floyd 2012c, and discussed further in Floyd forthcoming c.

specific action you are practising at). But that is not the sort of context at issue in this case. *With* an appropriate context in life, the rule *can* be followed; but *without* one, not. We have reached the limits of step-by-step procedures *in* formal logic, just as Turing wanted to show. They lie in everyday life.

By *not* building negation directly into his argument, then, Turing is doing something distinctive, making his argument more general. He thoroughly develops his analogy with the human being engaged in a calculating procedure. And what he does importantly differs from the now usual way of proving the negative resolution of the *Entscheidungsproblem* with the “Halting Argument”.⁴⁹ Here one constructs a “contrary” machine that negates the results of other machines as it goes down the diagonal, resulting in a contradiction. In this case, negation is built *in* to the machine. Turing’s argument does not proceed in this way, for the sake of generality and philosophical clarity.

In Turing’s way of making the argument, the notion of a general decision procedure for logic ultimately falters on two things: 1. an analysis of what a formal system *is* in terms of an *analogy* with a human computer, and 2. the construction of a rule that cannot be followed without supplementation of a context. Turing shows how logic cannot provide general definite methods of the kind Hilbert had sought, but from *within* pure logic, by means of logic alone. There is no theory of mind or of logic behind *this*. Only logic, as humans use it.

Turing’s paper must have struck Wittgenstein. After all, Wittgenstein himself had been the first to frame the idea of a decision procedure for all of logic in 1913, in a letter to Russell: this was *his* problem.⁵⁰ And the Hilbertian ideal expressed in the

⁴⁹ Martin Davis is presumed to have been the first to give this argument in 1952 (http://en.wikipedia.org/wiki/Halting_problem#History_of_the_halting_problem); cf. Copeland 2004, 40 n 61. Copeland (2004, 38) reconstructs Turing’s 1936/7 argument as a regress argument, but this is yet another rendition, as Floyd 2012c argues.

⁵⁰ WC, #30-37 (November or December 1913 to February 18, 1914), 56-69; cf. Dreben and Floyd 1991, 32.

Entscheidungsproblem, the idea of “calculation in a logic”, was in a sense a logical ideal of the *Tractatus*.⁵¹

There is a further point that supports, as I see it, an *a*-psychological reading of Turing’s great paper. Turing’s result is “absolute”. Gödel himself described this feature of the analysis as nothing short of a “miracle”:

It seems to me that [the] importance [of Turing’s analysis] is largely due to the fact that with this concept [computability] one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen. In all other cases treated previously, such as demonstrability or definability, one has been able to define [the fundamental notions] only relative to the given language, and for each individual language it is clear that the one thus obtained is not the one looked for. For the concept of computability, however, although it is merely a special kind of demonstrability or decidability, the situation is different. By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion.⁵²

Gödel insisted, in fact, that it was Turing who had established the general applicability of his own incompleteness theorems: before Turing had clarified the notion of a “formal system”, the scope of what Gödel’s results showed was just not clear.⁵³ But no theory of mind could have helped to accomplish this.

Of course, Turing’s analysis recaptured and exploited Gödelization, the possibility of coding the behavior of any formal system. For the “absoluteness” of Turing’s analysis is shown by the fact that just one “Universal” machine may, with a straightforward coding, do the work of all. The Universal machine offers a robust

⁵¹ The literature on whether or not the *Tractatus* was committed to a decision procedure for logic is longstanding. I believe it was not, though the situation must be rationally reconstructed, as Wittgenstein was hazy around the margins. Some arguments were given in Dreben and Floyd 1991 and Floyd 2001a, but more precise reconstructions were offered in Sundholm 1992 and Rogers and Wehmeier 2012. Weiss 2013 and forthcoming, so far as I am concerned, satisfactorily analyze the logic in the *Tractatus* as a predicative fragment of second order logic, pinning down the (very high) complexity of the definability of logical consequence given Wittgenstein’s basic framework.

⁵² See Gödel 1946/1990, 150, discussed in Floyd forthcoming c; Sieg 2006, especially pp. 472ff and Kennedy 2014.

⁵³ In Gödel 1986, 195 (a note added in 1963 to a reprinting of his famous 1931 incompleteness paper), Turing’s analysis was called “a precise and unquestionably adequate definition of the general notion of formal system”, one allowing a “completely general version” of his theorems to be proved.

characterization of the general idea of a “step” in a formal system, while demonstrating that the fluidity of simples in practice does not jeopardize the unity of logic.

Each individual Turing machine is given by a list of symbols and commands. Each individual machine has its “simples”. And yet each of these “beginnings” nevertheless finds its place in the one Universal machine, which in any case is (like all the other machines) set against the backdrop of our human procedures writing down step-by-step procedures for computing. Instead of mental states, one should think of a large insurance office with hundreds of people working at calculating machines, handing off results to one another throughout the day: one can imagine adding on further and further rooms and workers, horizontally and vertically, without ever meeting an intrinsic barrier to adding on further (except resources, interests, and purposes). The basic context is a social one.

Turing Machines, like people, are offloading platforms for calculative tasks. Discussion of theories of mind misses this point, whereas descriptions that respect variability, purposes, environments and needs do not. Turing showed, of course, that the Universal machine cannot generally survey and predict, of its own accord, the step-by-step behavior of all and every machine. But this is not, so to speak, because it is working on itself. Instead, the totality of computable functions cannot be diagonalized out of: it is not to be transcended in that particular way, in that “computable” is not the same as “true”. For in the general case there are *partial* functions, certain systems of definable commands that do not yield a result on a given input – there is “nonsense” in that sense. Each machine is capable of working on its own commands and states and outputs – including the Universal one, which thereby works on all commands, states, inputs and outputs. What the Universal machine implies, as Davis has argued (Forthcoming), is that there is no *trichotomy* between data (input, given experience), software (rule, calculation, inference) and hardware. Instead, the boundaries here too are fluid.

Wittgenstein would have seen and appreciated this much. Such appreciation would have given him faith in his own new ideal of

the fluidity of simplicity, as well as the fundamental power of his uses of language-games. The notion of a symbolically formulated, step-by-step procedure, so central to his conception of the logical since the *Tractatus*, had itself been logically, i.e., philosophically, analyzed. It would hold up.

The dream of an overarching calculus of logic, a well-founded picture of the world based on simples, or relative simples, or a series of simples, had to be surrendered: the *Entscheidungsproblem* was resolved in the negative. And yet, thanks to Turing, an absolute, self-standing conception of logic could be coupled with a primary emphasis on the fact that “common sense” (scrutinized endlessly, of course) would be required to fashion routines and allow us to understand and implement logic. Logic and our ideals of precision can and will survive this. But only by being transformed, situated in the realm of our ordinary discussions and procedures, to and fro, those harmonies among us as to purpose, point of view, and the sequencing of certain steps and procedures in our lives.

Toward the end of his life, partly under Wittgenstein’s influence, Turing *himself* came to emphasize these very points. His (1944/5) paper calling for a “Reform of Mathematical Notation and Phraseology” expresses his debt to Wittgenstein’s 1939 lectures (LFM) in helping him formulate a version of the theory of types. Centrally, he stressed here the need for everyday “common sense” “phraseology” as part and parcel of the development of logic.⁵⁴ In his report on “Intelligent Machinery” (1948/1992) he stressed the importance of “the cultural search”, by which he meant *human* cultural searching, with no machines mentioned as part of this, the individual regarded as a small portion of the whole society.⁵⁵ Turing’s famous “Computing Machinery and Intelligence” (1950) – which Wittgenstein seems never to have read – was a lighthearted joust at philosophers, and argues, not for mechanism in the

⁵⁴ Turing 1944/5, cf. my commentary Floyd 2013a and Floyd forthcoming c.

⁵⁵ Turing 1948/1992, second to last section of the report.

philosophy of mind, or genuine artificial intelligence, but instead that we *have* no fixed, general definition of “intelligence”.⁵⁶

Finally, in the very last paragraph of the very last paper he wrote Turing remarked that

These [limitative] results, and some other results of mathematical logic, may be regarded as going some way towards a demonstration, within mathematics itself, of the inadequacy of ‘reason’ unsupported by common sense (1954, p. 23).

Lebensform in Wittgenstein becomes a kind of replacement for this version of the wider Cambridge notion of “common sense”.⁵⁷ My point here is that this rigorization resonates well with Wittgenstein’s mature conception of philosophy.

4. Lebensform as a Logical Notion

4.1 The Idea of a “Chain” of Reasoning

I am arguing that while it occurs but five times in the *Investigations*, the timing of the initial occurrences of *Lebensform* in Wittgenstein’s manuscripts reflects an important gauge of Wittgenstein’s development. As I will explain below, this is reflected in the fact that his uses of the notion are progressively embedded in a wider flow of thinking about the concepts of *form* and *life* – most importantly, the notion of the *logical* – and finally then turned toward notions of certainty at work in the foundations of mathematics. In a broad sense, in Wittgenstein’s philosophy the accent seems to approach more and more to *life* as time goes on. But logic, with its ideas of *form*, *possibility*, *necessity*, and its (endemic) metaphor of *chains of remarks* (or reasoning) (cf. PI Preface), is never left behind as part of the quarry. Wittgenstein still conceived

⁵⁶ Wittgenstein wrote to Malcolm of Jan. 12, 1950 that he “had not read” the paper but “I would imagine it is *no* leg pull” (WC #429, 469). Malcolm had evidently suspected it might be. Gandy 2004, 125, reports that “I can remember [Turing] reading aloud to me some of the passages – always with a smile, sometimes with a giggle”, inferring that perhaps the paper was not meant to bear too much weight, but was instead to make “propaganda” for the philosophical significance of what Turing was doing with stored program computers. On there being no general definition of “intelligence” in Turing 1936/7, see Winston forthcoming.

⁵⁷ See Floyd forthcoming c.

himself to be pursuing the nature of the logical, conceived of what he is doing *as* logic, sees philosophy *as* logic, until the end.⁵⁸

This emphasis on form-as-logical has usually not been argued for in connection with Wittgenstein's uses of "*Lebensform*". Most readers have focussed narrowly – either on piecemeal "language-games" or the programming of our active behavior with words in terms of training in rules – or widely, thinking of *Lebensformen* as ways of life in the large, cultures, or organic internalizations of potential ways of acting in individuals who belong to certain cultures.⁵⁹ The former fits the stress on variety of differing snapshots or models of language use invoked in §§19 and 23; the latter seems invited by Wittgenstein's exploration of the concepts of *hope* and *grief* in PPF i §1, where he begins by contrasting a dog with a human being. But in either sort of reading Wittgenstein's association of *Lebensform* with scaffolding and logic, with binding and weaving, with handwriting, character, and certainties in our proceedings, including in mathematics (in PI §§240-242 and PPF xi §§345-362) is unfortunately lost. We need to unbury this cluster of associations in which *Lebensform* is embedded.

In general, the way to unbury it is to stress the centrality to Wittgenstein's life of the question, *What is the nature of the logical?* From his point of view it is logic alone, in reflection, that can get at this. One can surely read the PI as addressing itself, throughout, to this question. The same may be said of the TLP, of course, although the metaphors Wittgenstein uses there are more static, as befits his absolute notion of simplicity in that work.⁶⁰

As we have just seen, Turing's *philosophical* work recasts the question "What is a logical step?" insofar as what is meant is a *formal* system of logic. Turing brought that notion down to earth, domesticating it by drawing it into a world of human activity.

⁵⁸ See *On Certainty* [hereafter "OC"] *passim*; esp. at §§68, 82, 375, 501. I agree with recent work of Kuusela (manuscript) and Grève (forthcoming), who have developed the idea of the later Wittgenstein's work as engaged with logic. This conviction shaped earlier work of mine; for the thread concerned with philosophy of logic in my own work cf. Dreben and Floyd 1991, Floyd 1991, 2000, 2001b, 2005, 81), forthcoming a, b.

⁵⁹ Hunter 1968 forms a good starting point here, in distinguishing the narrower, wider, and "organic" picture of *Lebensformen*. Winch 1958/2008 is the *locus classicus* of the culture-wide idea, on which see Stern 2004.

⁶⁰ See Tejedor 2015 for discussion of the origin of *Lebensform* in the concept of *form* in TLP.

Turing was reported to have stressed this as a philosophical ideal already as an undergraduate, in a 1933 lecture to the Moral Sciences Club (very likely after he had begun reading, if not studying, Wittgenstein's philosophy):

[On 1 December 1933] A.M. Turing read a paper on 'Mathematics and Logic'. He suggested that a purely logistic view of mathematics was inadequate; and that mathematical propositions possessed a variety of interpretations, of which the logistic was merely one. Signed, R.B. Braithwaite.⁶¹

What is happening in PI is a further, reflexive, deepening domestication of just this idea. *Lebensform* lies at the ultimate reaches of this project, just as Turing himself later admitted. Wittgenstein deepens Turing's analogy with the step-by-step activity of a human calculator by situating this against the backdrop of, and embedding it within, the whirl of human life.

On this reading *Lebensformen* are "to be accepted" in the same sense that logic is "to be accepted". But how is that, according to the mature Wittgenstein?

From the outset we must insist on a very broad conception of logic, one not restricted to "logistic". As in Russell, as in Frege, "logic" should be understood in a primordial and encompassing sense. Logic *is* philosophy in Wittgenstein. As I shall use the term, it reflects activities that are irreducible, but everyday: the back and forth of argumentation, reasoning, reflection on such, and dialectic: Yes and No, but also here and there, correcting, pointing out, qualifying, precisifying, organizing and amalgamating and sharpening and dulling concepts, and so on. Given that the PI is filled with snapshots of such activity, this is one clear way in which it may be seen to be a book of, and an investigation of, logic, embedded in the philosophical tradition – critically, of course.

Wittgenstein concerns himself with forms of human life in which dialogue and procedures are imbricated; as his work goes on, these become situated in a ubiquitous, evolving, indefinite backdrop of contingencies, necessities, procedures, habits and possibilities that becomes ever more thematically rich and complex.

⁶¹ Hodges 1999, 6. I discuss this report at length in Floyd forthcoming c.

As he proceeds into his most mature writings, more and more is encompassed within the investigation of “logic” – as if his relentless quest to convey logic’s ubiquity in life did not end. This is the point of *Lebensformen*.

Now of course in Wittgenstein’s way of conceiving logic, arguments are not merely “sets of sentences” or “computable routines”, but instead specific structures that are put together, “chains of remarks” offered in some (living) context or other, whether with words, diagrams, letters, or a combination of these. One of his main points is that insofar as these are extracted or moved from their immediate homes, in life, philosophical questions about what the chain *is* are inevitably raised.

The image of a “chain” of reasoning may certainly be found in other philosophical writers. In logic, it is as old as the hills. It appeared in the *Prototractatus* and in the *Tractatus*, and was subsequently developed and explored throughout *Philosophical Remarks*, *The Big Typescript*, *Remarks on the Foundations of Mathematics* and elsewhere – as well as in the *Investigations*, where it is even used in the Preface to describe units of his mature style of writing. Significantly, as I have said, it is with an exploration of *this* image of a chain that TS 221, the second half of the *Frühfassung* of the *Investigations*, ends. It is as if Wittgenstein hadn’t quite worked that metaphor all the way through. For that, I would like to suggest, he would need *Lebensformen*.

As I shall put it on Wittgenstein’s behalf, *forms are linked, not ranked*. Logic in the primordial sense I am using it includes reasoning according to groups of considerations or beliefs that we have “chained” together into a certain order or structure, whether in ordinary language or formalizing.

Kant often uses the idea of a “chain” [*Kette*] to convey both regressive and progressive analysis, describing a “chain inference” as one “consisting of several inferences that are shortened or combined with one another for one conclusion”,

[For example,] Everything that thinks is simple, the soul thinks, hence it is simple. Everything that is simple is indivisible. The soul is simple, hence it is indivisible. What is indivisible is imperishable. The soul is indivisible, hence it is imperishable. What is imperishable persiss, the

soul is imperishable, hence the soul persists. – In this way I can make a whole chain of inferences, where I always make the conclusion into the premise of the following inference.⁶²

In fact Kant defines syllogistic method itself as “that according to which a science is expounded in a chain of inferences”,⁶³ and he treats “chains” of conditions, the “chain” of nature as an ordering of causes and effects, a “chain” of natural ends and necessities in the sensible world, the “chain” of mere associations in imagination, and so on. Reason itself has the need for intermediate links [*Zwischenglieder*] to help us orient ourselves in thinking.⁶⁴ Chains reflect necessities. Usage of the idea of a chain is common after Kant. Lewis Carroll, in his ironic book on Euclid, used it.⁶⁵ In *Principia Mathematica* Whitehead and Russell refer to “chains” of reasoning as they are presented as derivations from axioms, although they do not bring the notion of necessity into their use of it.⁶⁶

It is characteristic of Wittgenstein, thinking through the nature of the logical, to zero in on the image. In a sense he *has* to. It is a logical must.

The chain idea is of course to be found in Frege, bound up with his ideal of a “gap-free” [*luckenlose*] presentation of reasoning [*Schlusskette*]: chains of inference hold *together*, are a unity, allow us to move step by step through the structure. In stressing the nature of proof *in* logic, Frege wrote,

⁶² Kant 1992, Jäsche Logic, §8.

⁶³ Kant 1992, Jäsche Logic, §118.

⁶⁴ Kant 1781/1787/1998 *passim*; 1783/2002 4: 269, 333, 354; 1790/2000; 1790/2000 5: 298, 430, 435. At 1798/2002 7: 176 Kant writes that

[Psychological laws of association] often extends very far, and the power of imagination often goes so fast from the hundredth to the thousandth that it seems we have completely skipped over certain intermediate links [*Zwischenglieder*] in the chain of ideas, though we have merely not been aware of them. So we must often ask ourselves: “Where was I? Where did I start out in my conversation, and how did I reach this last point?” [Note: Therefore he who starts a social conversation must begin with what is near and present to him, and then gradually direct people’s attention to what is remote . . .]

⁶⁵ Dodgson 1879/2009. The image of a chain does not appear in Carroll’s famous paper on the regress of rule-application 1895. This is because it is being broken in this paper.

⁶⁶ Cf. Russell 1903/1938, Preface; Whitehead and Russell 1910/1927, Preface vii. In Summary of Part I, 87ff. he takes the work to possess two “aspects”, depending on two different points of view: “1. a deductive chain depending on the primitive propositions” and “2. a formal calculus”.

The length of a proof is not to be measured by a ruler. One can easily make a proof look short on paper by missing out many intermediate links in the chain of inference and letting much be merely indicated. One is generally satisfied if every step in the proof is obviously correct, and this is fine if one merely wants to be convinced of the truth of the proposition to be proved. If it is a question, however, of gaining an insight into the nature of this obviousness, this way of proceeding is not enough, but one must write down all intermediate steps, to let the full light of consciousness fall upon them.⁶⁷

Here, of course, Frege has an ideal of *completeness* in the logical representation of inference in mind: *fully* rational inference insofar as it is purely logical. He devised his *Begriffsschrift*, his presentation of the basic logical laws, because his requirement that we avoid gaps in argumentation is difficult: we need to be able to take reasoning *in*. In fact, Frege goes so far as to call it a requirement of reason that we must “be able to embrace all first principles in an *Übersicht*” (1884/1974, §5). The difficulty, he says, is

. . . the prolixity of a step by step approach. Every proof that is only slightly complicated threatens to become monstrously long. In addition, the enormous variety of logical forms revealed in ordinary language makes it difficult to delimit a set of modes of inference that covers all cases and is easy to survey [*leicht zu übersehen*].⁶⁸

As to the idea of a “step by step” approach, Frege analyzed the notion of the ancestral of a relation in terms of general logical laws governing concepts and objects, eliminating any appeal to intuition or an unanalyzed idea of an indefinitely iterated procedure. In this way he refashioned an earlier analysis that had been framed in terms of *Dedekind’s* idea of a chain (*Kette*). Dedekind had used *this* notion to analyze the very idea of a recursive specification of an infinite set. But he believed that his specification of the natural numbers turned, ultimately, on “the simplest act” of “successive”, step-by-step creation of one member of the series from another in thought, and his proof of the existence of such a “system” was based on an argument based on appeal to a thought of himself as

⁶⁷ Frege 1893/1903/2013, Preface VII; cf. Frege 1879/1967 III; 1884/1974 §§4,90; 1914/1979, 200.

⁶⁸ 1884/1974 §91.

an object.⁶⁹ Russell, persuaded in part by Wittgenstein take the axiom of infinity as an axiom, famously held that Dedekind's idea of "free creation" has "all the advantages of theft over honest toil" (1920, 271).

Wittgenstein is concerned with necessities of ordering. But in my view he does not revert to anything like an idea of "free creation" of these in PI. Instead, the dynamic, living character of *his* "chaining" of remarks is fashioned in order to investigate possibilities and necessities of "chaining" as such. This is what is distinctive about Wittgenstein's mature rendition of the idea of a chain of reasoning in PI. Such a chaining evinces *possibilities* and also *necessities* of movement and non-movement, shortening and parsing, rearrangement of links, the possibility of tracing from end to end, backwards and forward. It is a synthesis, holding a train of thought together, and also analyzable into parts – which may, however, be substituted into by other parts, other linkings and chainings, other syntheses, and so on.

For Wittgenstein the image is apposite in that logic, as he sees it, rests upon no metaphysical foundations of the whole, no "glue" of relationships, no prior ordering or features of the world, but instead upon our activities in fashioning links in our world, one by one. This does not mean that he reverts to the idea of will, or free creation, at the basis. Instead, we are binders, working against a universal backdrop, a whirl of life in which procedures, routines, chainings, have already been imbricated. Logic is concerned with establishing and interweaving procedures and possible procedures, as well as plumbing the limits of certain kinds of procedures. Insofar *it* can come to be indefinitely imbricated and impressed, ubiquitously, in lives with speech and language.

Now Wittgenstein always thought of logic as a matter of presenting agreements and disagreements (*Übereinstimmungen, nicht-Übereinstimmungen*) of possibilities and necessities, one with others,

⁶⁹ Dedekind 1888/1996, see 768. For analysis see Sieg and Schlimm 2005, Sieg 2013, 46, Floyd 2013b, 1023ff. Russell discussed Dedekind's notion at length in 1903/1938, so that Wittgenstein would have known it from there, at least. We do not know whether he ever read Dedekind directly.

others with others, and so on.⁷⁰ For him an articulate chain (later, a language-game, or a rule that is followed) draws out *aspects* of thoughts, given in fields of necessity, contingency, and possibility that we can take in; it does not (only) directly depict what is the case. Such aspects are erected, taken in, and then may be proceeded with, reconfigured, generalized on, and so on. The aspects are not merely “subjective”, even if they are response-dependent in a number of senses. Instead, they show themselves forth, are there to *be* seen, to be discovered – not directly, as objects are perceived, but in the way that a chain of reasoning may be taken in, seen, ordered, surveyed.⁷¹ Of course, Wittgenstein never *analyzed* possibility or necessity or reduced it to other terms. Instead, he takes us to be capable of “seeing” it in the workings of logic.

Now Frege too spoke of “aspects” in connection with logical analysis. He took generalization through reflection on function-argument structure to reveal “aspects” of thoughts, thereby emphasizing the organic, multiply interlocking character of his logic’s handling of generality in comparison to e.g. Boole’s. This is evinced in the multiple analyzability of, e.g., “Caesar crossed the Rubicon” into “*x* crossed the Rubicon”, “*x* crossed *y*”, “ $\Phi(\text{Caesar})$ ”, and so on, determining the structure of multiple generalizations that are possible.⁷²

But for Wittgenstein, unlike for Frege, such rephrasing draws in chains (orderings, fields) in which possibility and necessity show forth, come to life. Such chains or orderings of remarks allow for “intermediate” links among cases through comparison, substitution or rearrangement of parts (e.g., in form series; but more broadly on which see PI 122, 161): all the activities of ordering that make such a chain *Übersichtlich*, or surveyable. What are substituted are procedures, rather than, as in the pre-Frege tradition, terms, or, in

⁷⁰ Cf. TLP 2.22, 4.42, 4.4, discussed with a stress on modality in Shieh 2014 and Shieh manuscript. I discuss the first appearance of the idea of drawing “features” out of a “puzzle picture” in Wittgenstein’s wartime notebooks in Floyd 2010.

⁷¹ In Floyd forthcoming a I draw out the notion of “aspect” in Wittgenstein’s thought, parting ways with an aestheticizing and/or phenomenological sense of “aspects” as primarily experiential or “subjective” (cf. Floyd 2000, Baz 2000, 2010, Mulhall 1990, 2010). I think aspect should be characterized, like “showing” in terms of the grammatical “middle voice”, discussed in Narboux 2014: it spans action and passion, response and imposition, showing itself *in* our activities.

⁷² Frege 1891/1997, 5; 1892/1997, 27.

Frege, ways of generalizing over concepts and objects. In Wittgenstein's later philosophy all this takes place against an explicitly living backdrop – after 1937 *Lebensformen* – which are themselves brought into Wittgenstein's remarks by the imagination, the form of life, the logical and reflective capacities of the reader (as logic was also in the activity of the reader of TLP, though there there was not yet a significant role for *Lebensformen* to play).

Thus even in later Wittgenstein “logic” does comprise iterative, repeatable, reproducible procedures, routines that begin and terminate, that are well-founded (in that they begin *somewhere*), that are designed for a purpose, that may be broken off, amalgamated, communicated, interwoven, and shared. There *is* simplicity. Logic is still “formal”, without being “formalist”. Frege was right to criticize the formalists, from Wittgenstein's point of view. But he was wrong to equate the form/content distinction with that between sign and signified.⁷³ Wittgenstein “logicizes” Frege by way of his own organic notion of an *aspect*, using *it* to reconceive of multiple generality in terms of step-by-step procedures framed by language in logic.⁷⁴ The notion of *aspect*, like that of *form*, was always a *modal* notion, an idea of ordered necessity of reasoning in Wittgenstein, and in PI it becomes morphologized, living.

Wittgenstein's turn toward *Lebensform*, possibilities of life-structuring in life, would only have been reinforced by Turing's drawing a comparison between human computational activity and the idea of a machine blueprint into the foundations of formal logic. But it pushes deeper into philosophy itself, into the project of exploring and expressing *logos* in general in *logos*. In his discussion of aspects Wittgenstein draws in and presses forward what I have elsewhere called his “Master Simile”, a simile that also occurred in the TLP: a comparison between *logical* features of what we say (“internal” or “formal” relations among these) and *facial* features, aspects of a human face that wears a certain *look* (happiness, grief, etc.) and so is more than just the set, or sum, of its elements.⁷⁵ Such

⁷³ Cf. Frege 1891/1997, Mühlhölzer 2008.

⁷⁴ Cf. Frege's use of aspect in 1891/1997, 5; 1892/1997, 27.

⁷⁵ TLP 4.1221. I discuss this simile, with attention to Wittgenstein's development, in Floyd forthcoming a, building on Diamond 1991 and Putnam 1999 lecture 3.

aspects highlight or bring to life the relations a face bears to other looks and faces, i.e., it opens up a field of procedural necessities and possibilities of comparison, agreement, disagreement, similarity, difference, and so on.

Wittgenstein took this Master Simile very seriously throughout his life. And in PI it serves to deepen and extend Turing's snapshot comparison. Its methodological import is that in order to become *acquainted* with logical, formal features of what we say and think, we must work by looking, responding, querying, comparing, and so on – just as we do with a living human face (or even a picture-face) in life itself.

Wittgenstein's notion of acquaintance, of recognition [*erkennen*], is an everyday notion, woven into the notions of aspect and of familiarity; it is nothing less than the sense in which, by looking, listening, probing, discussing and responding, we become acquainted with a *particular* person, emotion, proof, perspective, look, and so on. This is Wittgenstein's altogether worthy replacement for Russell's alternative construal of the notions of acquaintance and aspect lying at the bottom of our appreciation of logical form. In Russell we have direct, incorrigible mental contact with simples, with aspects of things, with our own selves, with logical forms. In Wittgenstein such acquaintance only appears with coaxing, reflection, procedure, listening, and discussion: with methodical, dynamic, concerted interaction and a drawing out of spaces of contrast, of possibilities.⁷⁶

Life and reality are to be appreciated in linkings of aspects and in reshuffling of simples, philosophy in "chains of remarks" that may be broken off, revisited, resituated. In this sense the problem of other minds is far deeper than that of the external world, as Cavell argued long ago (1979/1999). But, in the end, perhaps the one problem mirrors the other, analogically speaking, as both are reduced to the problem of ordering and chaining in life. We bind ourselves to ourselves and to one another, in thought, deed and word. We bind ourselves to routines, to words with words. When Wittgenstein wrote that all knowledge in the end bases itself on

⁷⁶ For an extended treatment of this see Floyd forthcoming a.

(*gründet sich*) acknowledgment (*Anerkennung*), he was still, after all, echoing Frege.⁷⁷

In what follows I will run through the five occurrences of *Lebensform* in PI with these thoughts in mind. To keep my reading of Turing in play, I will keep philosophy of mind very much to the side. For me, the second, rougher part of PI will always remain “Part II”, though I am aware that it has been renamed “Philosophy of Psychology: A Fragment” in the new 2009 edition, and I concede to “PPF” referencing and section divisions below. Neither Turing nor Wittgenstein are given proper credit for being philosophers of logic, first and foremost, it seems to me. To be sure, they were both concerned to extrude mysterious, inner mental acts from the foundations; like Frege, Wittgenstein was always on guard, when a difficult question is asked about what something is, not to invent a something, anything you please, to fill the role. But then philosophy of mind is an *application*, and less fundamental to what they did, than their recasting of the notion of logic.

4.2 PI 19: Form of Life and Logical Aspects, not Culture and not Grammar

PI 19. It is easy to imagine a language consisting only of orders and reports in battle. – Or a language consisting only of questions and expressions for answering Yes and No – and countless other things. – And to imagine a language means to imagine a form of life [*Lebensform*].

As part of our depiction of *Lebensform* as logical, it is important to stress that Wittgenstein’s use of this notion is intended to *replace* an undifferentiated notion of culture, broadening and deepening it. The notion of culture [*Kultur*] does not appear in PI at all. *Kultur* was expurgated already from the *Big Typescript* and the remarks on Frazer’s *Golden Bough* – surely something intentional. This expurgation is reflected in the progress of Wittgenstein’s thought we have characterized above. For the places where he still uses *culture* in *The Brown Book* and in RFM I are juxtaposed with remarks

⁷⁷ OC 378.

that use the notions of *logical proceeding* and/or *life* to clarify *those* uses. These remarks draw in broadly logical features of imagined or longstanding historical culture, i.e., “natural history” involving *recognized procedures*: differing uses of color terms than ours; calculation with figures in the sand, as drawn from the ancient paradigms of procedure in Babylonia; a procedure in which there is taken to be a definitive end to the series of “natural” numbers.⁷⁸

No particular culture figures in the *Investigations* except its own, as Cavell has said (1988). In “the darkness of this time” in which it was written, it wasn’t clear there was such. Following Cavell’s lead, we may say that the “culture of voice” in the *Investigations* explores the *possibilities* of culture, not the actual, or even an imagined, world. It is schematic, not *bourgeois*. It is concerned with the very idea of an attempt to lay the foundation for any *possible* state, but it does not, like the *bourgeois*, attempt to shore up *this* state.

Here I echo a remark Wittgenstein made already in 1931 about Ramsey: he labelled Ramsey a “bourgeois” thinker, because Ramsey insisted on shoring up the laws of classical logic in the face of what he called the “Bolshevik” menace of intuitionism (cf. Ramsey 1926).⁷⁹ Wittgenstein, by contrast, was interested in the foundation of any *possible* state, not the *actual* one. Here we must remember the generality of Turing’s analysis of a “step” in a formal system: it bypasses the argument between classical and non-classical logic. For it is no part of our *general* notion of following a rule in a system that we follow classical, as opposed to intuitionistic, rules, or that we use what *we* call the “natural” numbers, as opposed to “1, 2, many”.

Though Wittgenstein writes about *Kultur* in letters to acquaintances – and the notion was very important to his self-conception (as in the drafted Foreword printed with *Philosophical Remarks* (PR)) – his mature philosophy is pursued without reliance on, or presuppositions about, this. His unwillingness to so rely was

⁷⁸ BB, *Brown Book* §3, 134 RFM I 153, II 27 (1938).

⁷⁹ MS 112 70v, 1 November 1931. This remark occurs in the midst of very interesting discussions of the boundaries of mathematics (using an Euler diagram) and a quote from Lichtenberg: “Unsere ganze Philosophie ist Berichtigung des Sprachgebrauchs, also, die Berichtigung einer Philosophie, und zwar der allgemeinsten”. [Our whole philosophy is a correction of the use of language, therefore, a correction of a philosophy, that is, the most general one.]

already made quite explicit in his initial lecture at Cambridge in Michaelmas 1930. Stating that there has been a “kink” in the development of human thought, he held that the “style of thinking” has changed, the “nimbus” of greatness has been lost to philosophy since the time of Galileo, for a “method” has been found (he meant here science or, more generally, analysis). Now “for the first time” there can be “skillful” philosophers, but the price is a “general tendency” of the age to “take away possibilities of expression: which is characteristic of an age without a culture” (WCLM 5:2, p. 107-8). This, of course, reflects Wittgenstein’s middle, hybrid view of simplicity, in which the idea of an archipelago of calculi reigned ascendent.

But what is interesting is that even here he associates the presence of a “culture” with “*possibilities* of expression”. Not everything may be said or expressed or meant at any arbitrary time, but what would be central to the work of logic (of philosophy) are the modalities, the possibilities of possibilities, opening up possibilities of expression, and not the facts *per se*.

My point here is that the notion of *culture* cannot be used without loss to gloss Wittgenstein’s idea of *Lebensform*, it is very much the other way around. Moreover, when Wittgenstein says that to imagine a language is to imagine a form of life (PI 19) he is struggling to embed the notion of language itself in a wider purview, stressing the open-endedness of his conception.

As to “value”: *Wert* does not appear in the PI either, and “normative” only once, in the reminiscence about Wittgenstein’s conversation with Ramsey about the idea of logic as a “normative science” (PI 81). This expresses a debt to Ramsey’s pragmatic, Peircean philosophy of logic, not a criticism of the idea of normativity in logic *per se*.⁸⁰ But there is an important contrast between Ramsey and Wittgenstein at stake. Ramsey wanted to investigate the *actual* state of things, beliefs, and persons (he was a “*bourgeois*” thinker). Whereas Wittgenstein wanted to try to schematize and carry through an investigation of all of their *possible* orderings. For this reason, in connection with the “normativity” of

⁸⁰ Here I agree with Misak 2016.

logic, it is quite important to stress that *Lebensformen* do *not* have to do with “value” as such, or “truth” as such. These notions lie in what we say and do, but in a step-by-step procedure as such there is a certain neutrality. This is precisely why particular routines are not neutral, cannot be carved away from what we do: values show forth, are expressed, in that we follow them, argue over them, surrender them, and so on.

As I have put it elsewhere, aspects (visible and communicable, reproducible ranges of features (of form), in living arrangements (configurations that we have put together) are *orthogonal* to value.⁸¹ They may intersect with it, be bound to or interlocked or interwoven with value, and surely aspects, once fashioned and seen, may make claims on us, set us off on trains of inference, and so shape our perspectives. But *aspects* as such have a moment of schematicity, of possibility rather than actuality, just as logic does.⁸² In terms of the duck-rabbit figure (PPI xi 118): *If* it’s a rabbit, then *this* is an ear, and *that* is a nose, *that* is a rabbit’s eye, and so on and so forth: a whole world of internal (logical) relations comes to life as we embed the figure in our lives. But *if* it’s a duck, then *this* is a beak, and *that* is a duck’s eye, and so on and so forth. This is part of the role to be played by *Lebensformen*.

In Cavell’s *The Claim of Reason* (1979/1999) progress was made on the notion of *Lebensform*: in his hands it is neither treated narrowly as a “language-game” nor widely as a whole human backdrop of agreement holding anyway among those who speak, nor as a culture. Instead, Cavell leans on and develops the idea with more sophistication, subtlety, and depth. He associates the notion with interpretation, but also dailiness and criteria: shared and communicable aspects of our ordinary lives and typical procedures with words (criteria), and, most of all, with the “ordinary” understood as a field of philosophical departure and return, a twin of scepticism.

Cavell thereby brings the broad notion of “logic” back to the *Investigations* in the sense I am articulating it here, and I regard my

⁸¹ Floyd forthcoming a.

⁸² Compare WVC 116 on “value”.

historical situating of PI as an elaboration, rather than a correction, of his reading.⁸³ The “ordinary”, the *simple*, as I have called it here, is colorful and dynamic, contested and accepted, an open and ubiquitous field for life, for philosophy. Of course, it may be rejected, or felt as limited, and then, with our everyday procedures shunned, scepticism may raise its head. The need for an “outside” of logic as a quest for *Luftleben*, life’s breathing air, must return to its environment in utterance as well as procedure (cf. PI 102). What is thereby shown to fall to the side in Wittgenstein is a certain conception of unlogical “gas”, empty structures of air [*Luftgebäude*]. Cavell’s idea of a culture of “voicing” inscribed in the *Investigations* (further articulated as *Lebensform* in Laugier 2015) gives us the right orientation for addressing culture such as it is or might be from the perspective of PI. After all, the chains of remarks in PI are nothing but investigations of possible voicings of thought.

We might say that Wittgenstein’s “forms of life” appear in PI as voices, orchestrations of possible forms of life, or journeyings of voicing with aspects. To see *Lebensform* as logical is to regard the notion as a necessary outcome of thinking through the idea of what, in general, calculation-in-a-logic *is*, a proper and inevitable foundational setting, a logical gloss on the essence of essence, an analysis of logical analysis *as* such.

But now I imagine an objection: does not Wittgenstein *replace* the notion of logic (or “logical syntax”) with that of grammar? Is this not a *generalization* on his part, as it were getting *past* an obsession with “logic”?

“Grammar” is most often applied to particular expressions in PI, usually words, verbs, or brief phrases; a grammatical “investigation”. But it is also connected with the kind of object an object is, with possibilities of phenomena (PI 53, 90; cf. PPF xi 186). Getting to see this may involve us in analysis of “forms” of expression, Wittgenstein says – though there is no *final* analysis by the time of PI (PI 90-91). Grammar is evinced in an exploration of

⁸³ In this sense, Cavell’s reading of Wittgenstein is not “dark”, as is argued by Moyal-Sharrock forthcoming: it encompasses both light *and* dark “modernity”, is in that sense neutral, shows us logical certainty, necessity, as it is.

our criteria, which show us the place of a concept: possibilities of use, not the actualities of particular objects or uses *per se*.

In Wittgenstein, grammatical explorations are, in my broad sense, logical: they reveal aspects of thoughts, words, sentences, states of affairs, ourselves, and so on. For they open up fields of possibility and necessity that are made, in logic, *alive* to us. If there is ever anything going on analogous to therapy in the later Wittgenstein, it involves not simply liberation through talking *per se*, but this only in seeing and taking in and comparing and living with *real* possibilities and necessities of expressing, possible (imagined) modes or forms of life.

By using “grammar” Wittgenstein is making a particular sort of move in relation to the tradition. He is holding, not terribly novelly, that grammatical explorations are part of logic: Aristotle had said as much. But he is not *replacing* logic with “grammatical investigations” only, for their own sake. This idea is – unfortunately – encouraged by the finally published form of PI, which is unfinished. This form has lent to Wittgenstein’s philosophical legacy too great a focus on language as such, as if he was no longer really focussing on the logical. But by focussing on the everyday, by saying that to imagine a language is to imagine a *Lebensform*, he is nevertheless clarifying the nature of logic, as the context in which he framed his ideas makes clear.

After the mathematicization of logic in a language post-Peirce and Frege, mathematicians such as Hardy and logician-philosophers such as Carnap stressed that ordinary parlance is vague, mushy, unrigorous, a seat of endless and fruitless argumentation. Hardy and Littlewood called it “gas”, “provisional nonsense” that would be dispensed with in proper mathematics.⁸⁴ Carnap recommended the replacement of ordinary phraseology with the construction of formalized languages, axiomatic systems. But, as Turing argued in 1933, the idea of “logistic” is but *one* interpretation of mathematics among many, and not a privileged one. PI expunges the whole idea of “gas”, recovering the

⁸⁴ Cf. Littlewood 1926, Preface and Hardy 1929, 18, discussed by Wittgenstein and Turing in Lecture I of LFM. I treat the significance of these metaphors in Floyd forthcoming c.

“foundations” through analogies, the metaphors, exploration of nonsense, and so on. It is a philosophy built along lines Turing explored and explicitly discussed with Wittgenstein in LFM (see Lecture 1).

Wittgenstein wanted to emphasize to mathematicians, logicians and philosophers that how we use words in everyday life (including in ordinary mathematics, outside logic) is inseparable and not to be dismissed from foundational talk. A philosophical investigation of ordinary “phraseology” is not merely “decorative” or aesthetic, it is an essential part of the work.⁸⁵ We have seen Turing putting this idea into action in his analysis and resolution of the *Entscheidungsproblem*, as well as his early talk to the Moral Sciences club (see Braithwaite’s report, quoted above). Turing self-consciously *domesticated* the sublime picture of mathematics as a “miraculous machine” that Hardy had used to explain Hilbert’s idea of an *Entscheidungsproblem*, thereby bringing it down to earth.⁸⁶ (Gödel too held that to get the foundations right we had to go *outside* of mathematics, to philosophy, ploughing through everyday life – but this is another story.⁸⁷)

This is why much of Wittgenstein’s method turns on exploring what we are actually inclined to say about what we might possibly say or do (or not do). Yet his quarry is (the nature of) logic, not proper word-use, or particular concepts, or chatting *per se*, even if these too would receive re-thinking in light of his later philosophy. In PI the distinction between “prose” and “proof”, a characteristic of his “middle” period philosophy of mathematics, becomes permeable, though still able to be drawn locally, as he comes to see the ubiquity, complexity and fluidity of reflection, both on and in “prose” and “proof”.⁸⁸ In the handling of “first” steps in the early

⁸⁵ This is the explicit topic discussed by Wittgenstein and Turing in lecture I of LFM; cf. Floyd forthcoming c.

⁸⁶ Hardy 1929, 18.

⁸⁷ Floyd and Kanamori 2016 outline some of Gödel’s 1942-3 *philosophical* reactions to Wittgenstein and Russell.

⁸⁸ Kienzler and Grève 2016 helpfully note the permeability of the distinction between “prose” and “proof” in Wittgenstein’s later writings. However they ground their interpretation of Wittgenstein’s 1937 remarks on Gödel on the evolution of Wittgenstein’s thought about language *per se* (cf. pp. 85ff). This leaves philosophy of logic out of the picture. Moreover, they attribute

version of PI (FF) the manuscript ends with the image of a chain, still not satisfactorily worked through. This working through would preoccupy Wittgenstein through 1939-40, when he was writing RFM III – manuscripts produced, not accidentally I think, after Turing had attended his Cambridge lectures on the foundations of mathematics in the spring of 1939 (LFM).⁸⁹

4.3. PI 23-25: Natural History and Logic, Regressive Analysis, Concatenation

PI 23. But how many kinds of sentence are there? Say assertion, question and command? – There are *countless* kinds; countless different kinds of use of all the things we call “signs”, “words”, “sentences”. And this diversity is not something fixed, given once for all; but new types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten. (We can get a *rough picture* of this from the changes in mathematics.)

The word “language-*game*” is used here to emphasize the fact that the *speaking* of language is part of an activity, or of a form of life.

Consider the variety of language-games in the following examples, and in others:

- Giving orders, and acting on them ---
- Describing an object by its appearance, or by its measurements –
- Constructing an object from a description (a drawing) –
- Reporting an event –
- Speculating about the event –
- Forming and testing a hypothesis –
- Presenting the results of an experiment in tables and diagrams –
- Making up a story; and reading one –
- Acting in a play –
- Singing rounds –
- Guessing riddles –
- Cracking a joke; telling one –
- Solving a problem in applied arithmetic –
- Translating from one language into another –
- Requesting, thanking, cursing, greeting, praying.

philosophical views to Gödel that, I believe, are in error, and misdescribe the “situation” in “ordinary mathematics” at the time (cf. Kanamori and Floyd 2016, Floyd forthcoming c).

⁸⁹ On RFM III see the magisterial commentary Mühlhölzer 2010, reviewed in Floyd 2015. Floyd forthcoming c comments on Wittgenstein and Turing’s discussions in LFM.

It is interesting to compare the diversity of the tools of language and of the ways they are used, the diversity of kinds of word and sentence, with what logicians have said about the structure of language. (This includes the author of the *Tractatus Logico-Philosophicus*.) ...

PI 25. It is sometimes said: animals do not talk because they lack the mental abilities. And this means: “They do not think, and that is why they do not talk.” But – they simply do not talk. Or better: they do not use language – if we disregard the most primitive forms of language. – Giving orders, asking questions, telling stories, having a chat, are as much a part of our natural history as walking, eating, drinking, playing.

Wittgenstein counts “chatting” [*plauschen*] explicitly among those most pervasive elements of our “natural history” in the *Investigations* (PI 25): this is the culmination of the first occurrences of *Lebensform* (in PI 19, 23). These take off from the metaphor of the city as an evolving organized whole, its “downtown” heart centered on winding, ancient parts rather than straight-running superhighways (PI 18). The PI is not just chatting, however. It comprises *argumentation*, a lot of it – something I have found those who write about Wittgenstein’s philosophical “methods” too often underemphasizing – as if they are primarily directed at the nature of proper language use and meaning.

There is a difference between chatting, requesting, insisting, and so on, and arguing and genuinely reflecting on *that*. Part of Wittgenstein’s philosophical aim is to teach us to characterize, negotiate, recognize, and utilize in our own ways these complexes of differences rightly. Philosophy turns on nothing more, and nothing less, than developing an ear, a nose, and a pair of eyes for the difference between chatting and arguing, questioning and supporting, qualifying and stating, analogizing and speaking literally, negating, and misunderstanding. All of this belongs to the activity of *logic*, as do all the procedural elements given in the list of language-games given in PI 23.

Wittgenstein suggests that we can get a “*rough picture*” of all of this from looking at changes in mathematics (PI 23). We have already broached this idea in looking at Turing. But perhaps a wider look around will help. In the development of mathematics we may

see certain earmarks of the conception of the logical that they are forwarding.

The conception springs from the very idea of analysis itself, as embodied in the “regressive” method of presenting a body of mathematical body of knowledge in axiomatic form. Here a certain dialectic or tension always existed between the way a proof is presented, and its *raison d’être*. This is very ancient, a living character of the axiomatic tradition in mathematics as well as the analytic tradition in philosophy. It inheres in the image of a chain of reasoning, which may be traced forwards or backwards, considered as a synthesis or as an analysis into parts.

Book I of Euclid’s *Elements* serves to illustrate. (Wittgenstein wrote much about Euclid.⁹⁰) It ends at Proposition 47, the Pythagorean Theorem. That is, in a certain sense, the *raison d’être* or goal of Book I. But the presentation, as so often in mathematics, is written *backwards*. The method of presentation is: *Given* this, you can get the theorem. *Given* that, you can get this. And so on. The moves are from synthesis to analysis and again back to synthesis.

Proofs, analyses, regressions, reductions to givens, are written *backwards*. So Hobbes read Book I of Euclid’s *Elements*, if we can believe Aubrey’s *Lives*.⁹¹ There is a systematic “backwardness” to be taken in. The “givens” are the beginning and the end, the interlinked chain as a whole, and also the particular linkings. *Necessity* appears, until it ends somewhere (*not* in self-evident “givens” or “aspects of acquaintance” in Russell’s sense, according to Wittgenstein). *Sufficiency* appears to work forward: *Given* this, you can get that; *Given* that, you can get the next, and so on. But we use this, in fact, to stitch our way *back* to the “givens” or simples. Euclid says that the “givens” are “data”, the “givens” at the front.

⁹⁰ For example, see BT pp. 105ff, 325-334. For discussion see Floyd 1995, 2000, Floyd and Mühlhölzer manuscript.

⁹¹ Stillwell 2010, 2.1, 18, quoting Aubrey 1680/1898:

He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman’s Library, Euclid’s *Elements* lay open, and ’twas the 47 El. libri I. He read the Proposition. By G – sayd he (he would now and then sweare an emphaticall Oath by way of emphasis) this is impossible! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read . . . that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.

But that is only a metaphorical way of thinking about the procedure.

Reading forwards, the uninitiated reader may be told once, at the outset, “This is where we are headed: here is the goal of the journey”. She may be reminded of this from time to time. But it is the journey, not the destination that counts. We mostly do not see, moving through as readers, where we are. The end is not yet chained, bound, in a series of procedures we can take in. The connection to the end may appear to lie somewhere underground in hidden machinery.⁹² Maybe we will never see the whole as a chain (a chain is visible, surveyable, perhaps through winding it up, running through it, link by link). Perhaps a piece of computational machinery, to which we offload, can help us to take it all in, archive and arrange it all, to see, take in, and form the necessities and possibilities in another way. Information requires formation, after all. But algorithms are not neutral: they inhere in, shape, swim, in *our* world of procedures, as Wittgenstein and Turing stressed.

Of course, these images of “forwards” and “backwards” are insufficient, ultimately metaphorical. A chain has no direction or orientation (unlike, say, an arrow or a pointing finger). In the *Elements* Book I, the aim is the composition and activity and arrangements, the chains of aspects themselves. The important thing is the possibility of *this*. By means of it, the Pythagorean Theorem is not merely a statement or headline. It is transformed into a living face, mathematics with *depth*. In writings generated in 1939-40 – just after Turing had sat in on his Cambridge lectures LFM – Wittgenstein remarked that mathematics is a “colorful mix”, likening it to a tapestry that we weave with filaments (of procedures), rather than a chaotic, unsystematic thing.⁹³ Moreover, he emphasizes that we *discover* aspects, whereas we *invent* techniques.⁹⁴ We might say that aspects *come out* of our framing of procedures (of proof or calculation).

⁹² As in RFM I 106, App. II 3; cf. Floyd 1995, 2010, 2012a.

⁹³ RFM III 46-48. The German is “*buntes Gemisch*”. I follow Mühlhölzer 2010 in rejecting the Anscombe original translation of “motley”, which makes it sound like a jester’s costume. Cf. Floyd 2015, 262.

⁹⁴ This contrast appeared before 1939, but is returned to in the later writings. Compare BT 134, RFM III 46ff, PI 119, 124-9, 133, 222, 262, 387, 536, PPF 130 and Floyd forthcoming a.

We are forwarding a picture of the PI as belonging to logic in the ancient sense: to philosophy, demanding as it does the representation of *logos* itself. From this panoramic point of view, the PI's way of thinking must be self-standing and alone competent to judge its own workings, as well as fluid and ubiquitous, embeddable everywhere: an endlessly interesting, endlessly articulable way of aiming to pursue – and also learn how to break off from – certain kinds of perfection, ordering, in thinking. Wittgenstein offers us an *orchestration* of voice: exemplary arrangements and linkings of forms or possibilities of voicings, harmonies, disharmonies, agreements and disagreements. These are compositions as opposed simply to chats, although chatting there is and – one would like to say – so it has been and ever must be.

There is a dim but distinct echo here of certain *Tractarian* ideas, but now subsumed into the mature way of thinking about the simples. This is why Wittgenstein practices his mature method of comparing his mature with his earlier views. He has not left behind or contradicted the idea that there is no special subject matter of logic. Logical signs still offer punctuation marks for possible thoughts, not reference to the constituents of senses (TLP 5.4611). What there are *in* logic are chains of interlockings of *possible* thought-trains. And logic's conception of thought is itself a conception of expressions of agreement and disagreement with possibilities, though now the truth-table way of clarifying this is no longer privileged.

As reflection and argumentation and chatting are pursued, they gradually assume the dress of further articulation. Such belongs to our “natural history”. As we know, “logic” is eventually explored in the PI in relation to the imposition and concocting of possible procedures or routines, and, more specifically, “calculi”.⁹⁵ Such was

⁹⁵ Pichler draws on a deep understanding of Wittgenstein Nachlass when he emphasizes (2016) that the “calculus” and “human” conceptions of language use vied with one another in a complicated, interwoven manner throughout Wittgenstein's life, one sometimes ascendant, then the other, neither wholly supporting or occluding the other. I am proposing that “forms” partake of both aspects. This reading is at odds with a straight-line progressivist interpretation on which Wittgenstein replaces a “calculus” conception of meaning derived from Frege, Russell and the *Tractatus* with an anthropological or ethnological conception, of which an open-ended, interlocking series of language-games, conceived as constituting “forms of life” or “cultures”, are the expression (cf. Hacker 2015).

logic's own history of formalization, our natural history with it. Let us consider that next.

In a quite ordinary sense the notions of “juxtaposition” and “concatenation” – “chaining” – are ancient and fundamental to the primordial idea of logic I have characterized above. Certainly such phenomena of chaining are strewn across the history of mathematics, of logic, of alphabets and writing. Philosophers recognized this long ago, beginning at least with Plato, who worried about the suppression of human voice in the face of writing. In TLP the idea of an elementary proposition, construed by Wittgenstein in terms of an articulate picture used to say how things are, embeds the idea. Wittgenstein's idea of ultimate simplicity, the basis of the space of all possible things to be said, turns on the idea of simple objects configured in chain, fixed in a determinate ordering as an articulate whole, a *tableau vivant* fixed by concatenation of symbols (TLP 2.03, 4.22, 4.0311). Wittgenstein has to work hard in the TLP to get past the use of a picture to say how things are, to get on to the idea that the same picture may be used to say how things are and how they are not, so that the use of the picture itself can bring to life logical form, a possibility of presenting alternative possibilities, ultimately within the whole world of possibilities as such.

The Babylonians appear initially to have put “7” next to “5” in *juxtaposition* to “add” and get twelve, a kind of fusion of constituents in which order is irrelevant (twelve sheep are “made up of” seven and another five). That was surely a procedure (so is “one, two, many”). Yet getting to the very *notion* of *concatenation* (“chaining” or “stringing”), essential for getting in turn to *routinization* and iterable *calculation*, and, further *indefinite iteration*, took large steps. The Babylonians were able to get to an alphabetical system (after some centuries) with their base 60 notation, thereby folding in what belongs in general to writing and ordering in the sense fundamental for any civilization, logical proceduring, or language. Concatenation draws in, *is*, ordering: perhaps first a bidirectional linear order of symbols, and then, later on, other forms of linear and partial order. The decimal notation, and “+”, came much later on, as did later still the more and more

general combinatorial issues (of partitions of numbers, of possible arrangements of sheep and officers of different ranks, of a system of recursion equations, of the ordered pair, and so on). As Otto Neugebauer, the great historian of ancient mathematics with whom Wittgenstein shared a pencil while in Monte Cassino once wrote,

What is often overlooked and cannot be sufficiently emphasized is the terrible difficulty and slowness of the development of the very simplest fundamental mathematical concepts, first of all of a genuine computational technique.⁹⁶

It is a remarkable fact that the regressive method itself only rendered a clear notion of *set* (unordered elements put together, *ensembles*, as the French have it), thereby reintroducing the ancient idea of static joining, so late in the game – in fact during Wittgenstein’s early lifetime. Here is a “given” *gotten to* through the regressive method, then capable of synthesizing an enormous playing field for mathematics in set theory. It was not one Wittgenstein particularly liked playing in, admittedly. But we can see why. For it obliterates, in its very nature, the aspectual, chain-like level of conceiving *procedures*, *putting together in orderings*. Thus it obliterates Wittgenstein’s central *data* of logic, just as it had for Frege before him. It also invites in by the back door the whiff of both absolute and relative conceptions of simplicity, a one-sided, undynamic picture, in which forms, once again, are ranked and not merely linked.⁹⁷

Once *concatenation* is in place, eventually notions of *substitution* and *occurrences of a sign* require hard thinking through in logic. As a matter of fact, these were the very issues with which Wittgenstein and Russell struggled in 1913.⁹⁸ Substitution was only formulated as a separate principle of logic in the twentieth century – a crucial preliminary to Gödel’s 1931 paper on incompleteness and also

⁹⁶ 1935-37, vol 3, 80. On Wittgenstein and Neugebauer see cf. Swerdlow 1993. I am indebted here to Kanamori (manuscript) for further discussion of the long history of proof. For Wittgenstein on Euler’s thirty-six officers problem and the bridges of Königsberg, see Floyd 2012a, 240.

⁹⁷ On Wittgenstein’s crabby, but not utterly useless remarks on set theory, see Floyd and Mühlhölzer manuscript.

⁹⁸ Cf. Russell 1913/1984.

Turing’s “On computable numbers” (1936/7).⁹⁹ Each of these “chainings” instituted new “forms of life”, and created and grew newfangled calculative behaviors on the part of humans, at the individual and the civilizational, cultural levels.¹⁰⁰

4.4 PI 240-242: *Agreement in Judgments, Scaffolding*

PI 240. Disputes do not break out (among mathematicians, say) over the question of whether or not a rule has been followed. People don’t come to blows over it, for example. This belongs to the scaffolding [*Gerüst*] from which our language operates (for example, yields descriptions).

241. “So you are saying that human agreement decides what is true and what is false?”

– What is true or false is what human beings say; and it is in their language that human beings agree. This is agreement [*Übereinstimmung*] not in opinions, but rather in form of life [*Lebensform*].

242. It is not only agreement in definitions, but also (odd as it may sound) agreement in judgements that is required for communication by means of language. This seems to abolish logic, but does not do so.
– It is one thing to describe methods of measurement, and another to obtain and state results of measurement. But what we call “measuring” is in part determined by a certain constancy in results of measurement.

I take it that with the image of “scaffolding” Wittgenstein is revising Hilbert’s idea of logic as part of the *Fachwerk* (half-timber structure) of a theory¹⁰¹ by making it *dynamic* in our varieties of its uses.

Hilbert had written to Frege that

⁹⁹ C.I. Lewis identified substitution as a separate logical principle in his 1918. Sanford Shieh has suggested to me that Lewis may have derived the idea of it from Royce, or, one might add, it may have come that way through Peirce. The use of “the method of substitution” in a broader sense is of course mentioned in the TLP 6.24, and has a long history in mathematics.

¹⁰⁰ As Wittgenstein argues, the notion of “following a rule” involves customs, institutions, *Praxis* (PI 7, 24, 51, 197, 202). These are at *heart* routines that are shareable – just as Turing’s “machines”, with their commands are.

¹⁰¹ Cf. Sieg 2013, 24n for historical contextualization and discussion.

Any theory is evidently only a *Fachwerk* of concepts or a schema of concepts together with their necessary relations to each other and the basic elements can be thought in arbitrary ways . . .¹⁰²

This evidently did not move Frege, who objected to Hilbert's schematic picture of logic, and his insistence that logical consistency of a theory in mathematics is enough. But for us it is important to note that the image of *Fachwerk*, however picturesque as a memento of Göttingen, is static, structural, holding a house together: Hilbert wanted an image of complete organization of a theory by means of its logical analysis and axiomatization. Wittgenstein counters with the idea of logic in *motion*: scaffolding that plays no structural role, and can be modularized, taken down and put back up again as needed. In the end, it disappears. This suits the fluidity of simplicity, as well as Turing's idea of proceduralization.

On Wittgenstein's view, logic draws aspects out of characterizations, it characterizes characterization itself, and discusses and reflects on this (i.e., on logic itself). "Same" and "different" are fundamental notions here, as are the notions of a "character" and "character" and order, handwriting with orientations of letters, specific arrangements. The notion of hanging together as a whole, while being step-by-step configured, link- or band-wise, is held together in a strong, specific, but transportable structure that is contingent, as a matter of fact, but gives rise to step-by-step necessities, once it is configured. Simplicity is not simple, but it *is*: a mixture of considerations, ontological and epistemological, an ideal of method.¹⁰³ Wittgenstein works hard on orientations of letters, even mentioning handwriting as a mode of *Lebensform* in the PI at PPF i 1.

Turing's analysis is impervious to what the specific signs might be in a particular computational routine. But a Turing machine has *two* faces: one static (a set of quintuples), one dynamic (the moving action of a machine given its commands). Turing has perfectly interwoven, in general, the interstices between juxtaposition and

¹⁰² Hilbert to Frege 29 December 1899, Frege 1980, 42; "scaffolding" is used to translate *Fachwerk*, but "half-timbering" would be better.

¹⁰³ Cf. Floyd forthcoming b.

concatenation, without confusing them, has kept in play the dynamic *and* the static, the complex *and* the simple, placing them all into one self-standing space, sitting within our world of “ordinary phraseology”. For the list of quintuples that define one of his “machines” are a *set* of commands that may be listed in any order (the concatenated ordering *within* each command is taken for granted). But what such a “machine” *does* is to concatenate dynamic possibilities of step-by-step procedures, moving forward.

The problem responded to is as ancient as Socrates’s dream in the *Theaetetus*.¹⁰⁴ It is thematized as well in Aristotle’s mocking of the atomists in *Metaphysics*, who “lazily neglected” to face the issue of *movement* in accounting for the simples, thinking that the difference between AN and NA, or H and Ξ could be accounted for in their terms alone.¹⁰⁵

It is our agreements in *ordering* that matter to logic. It is more generally agreement in measuring, weighing, gauging, voicing, and so on that is the harmony, the *Übereinstimmung*, of which Wittgenstein writes in PI 241. It is on this that not only the having of opinions, but even the very idea of “calculation in a logic”, *logos* itself, rigorously depends.

The logical notions of “concept” and “object” sit rather uncomfortably with Wittgenstein’s aspect-thinking, as neither extensional nor intensional ways of construing concepts fit directly into its way of articulating logic in terms of procedures and iterable operations. In PI Wittgenstein *had* to stress the importance of inventing “intermediate links”, because the idea of a totality or extension occludes his philosophical treatment of logic. Procedures, with our modal and dynamic ways of visualizing, applying and imagining them, are a more natural fit with the fluidity of simples than are either the notions of concept or object, except when these are handled comparatively, as they are within his language-games (forms of life).

Logic as Wittgenstein conceived it has to do, I am arguing, with investigations of *possible* juxtapositions, orderings, concatenations

¹⁰⁴ Cf. Parikh and Renero forthcoming.

¹⁰⁵ *Metaphysics* I 985 15, in Aristotle 1984.

(chains), iterable procedures, rules and rule-following. This is why the notion of *aspect*, as opposed to *concept*, is primary for him. Concepts, rules for the uses of words, and so on are understood in terms of aspects, not the other way around; the family resemblance idea is *necessary* for Wittgenstein, not simply a picturesque idea of generality or historicity inheritance from Goethe and Spengler: he has *logicized* morphology, morphologized logicism. The idea of family resemblance is logically *robust*: like that of a chain or interwoven thread, as Wittgenstein says. It is mode-like, *modular*, in its very essence, but it is alive and evolving in comparisons we make and act upon.

Felix Mühlhölzer has remarked, in discussing Wittgenstein's later philosophy of mathematics, that it has a "dark" side: in his later writings, Wittgenstein rarely if ever discusses axiomatic method. That's right, but there's a reason for this. Like Turing – and well before he met Turing, when he wrote the *Tractatus* – Wittgenstein did not reduce logic to the axiomatic method. Instead, he reduced that method itself to the terms of more general and fundamental ideas: rule formulation, catenation, iteration, substitution, and calculation: procedures of calculation that can be taken in, begin somewhere, and may be carried on, reorganized, and articulated indefinitely.

What Turing did in "On computable numbers" was to logically and mathematically rigorize Wittgenstein's attempted philosophical generalization on the axiomatic method. For, partly indebted to Wittgenstein, he rooted his analysis of the notion of "calculation in a logic", a "formal system" in a "language game", a snapshot of human calculative behavior, a person with pencil and paper living in a social world. Turing *analyzed* the logical aspect of our notion of an "axiomatic" or "formal" system by thinking through what we *use* such a thing *for*, by what is *done* with it.

Mathematicians no longer "come to blows" over whether or not a rule has been followed, in Turing's sense. There is no *need* to. There is, rather, the matter of assessing axiomatizations. As Wittgenstein says, *judgment* is required at the basis to determine which ones to embed where, and why, and for what purpose. And this too depends upon regularities of response, and our shared,

agreed upon proceedings with rules. In this sense the distinction between a calculation and an experiment – always an earmark of Wittgenstein’s discussions of mathematics – is domesticated, brought down to earth. Objectivity is not a concept everywhere bounded or constituted by sharp rules or a system of sharp rules. It cannot be. But for all that, we still can and do weave scaffolding capable of allowing us to proceed, and not come to blows. All this, just as Turing said, requires “common sense”, though in PI this notion is made dynamic, embedded in regularities of world and our human forms of life within it.

While Turing was an undergraduate Wittgenstein had thematized, with “language games”, the idea of a possible set of orders or commands and possible movements and reactions, possible impositions of procedures and possible steps taken with these by human beings living an active, shared life in words. The variety is most important, and when Wittgenstein discusses *Lebensform* at the beginning of the PI it is to stress just this. But this is not done with the aim of debunking the axiomatic method *per se*, much less logic itself, as if providing it with some kind of philosophical rival – for it has none. Turing’s own work can be axiomatized.¹⁰⁶

Instead, Wittgenstein wanted to probe down into the fundamental interstices *of* logic. “To imagine a language is to imagine a form of life” (PI 19), that is, a possible structuring of life, one that shows an aspect of life, draws out procedures we have and concoct *in* life. Echoing that key remark of the *Tractatus*: form is the *possibility* of (living) structured, proceduralized activity. The end of that, however, as Turing wrote (1954, quoted above), is the need for “common sense”.

Wittgenstein always wanted to show, among many other things, that logic has no basic propositions (in the sense of Euclid), requires no essential appeal to “self-evident” axioms or laws. It is not to be conceived as a science of thoughts or propositions in Frege’s or Russell’s senses, but rather as an activity in which we show how to construct apparent sentences and chains of reasoning

¹⁰⁶ Sieg and Mundici forthcoming.

that are tautologies and/or contradictions, in which we show how – as we might colorfully put it – to get to the limit of “therefore”, with no premises needed at all, just a “form” of pure possibility leading to nothing but an indefinite substitutability of words. We get to operations in which the force of *particular* word choices is made to disappear, so long as one substitutes in accordance with an ordered, fixed, procedure. What are fashioned and recognized are spaces of possibility, necessary interrelations among thoughts *through* interrelations among possible procedures. *Faces* of necessity. The scaffolding of what is true or false *in* what we say.

4.5 PPF i, 1-2: Übersichtlichkeit in the Tapestry of Life (Lebensteppich)

PPF i, 1-2:

1. One can imagine an animal angry, fearful, sad, joyful, startled. But hopeful? And why not?

A dog believes his master is at the door. But can he also believe that his master will come the day after tomorrow? And *what* can he not do here? – How do I do it? – What answer am I supposed to give to this?

Can only those hope who can talk? Only those who have mastered the use of a language. That is to say, the manifestations of hope are modifications of this complicated form of life. (If a concept points to a characteristic of human handwriting, it has no application to beings that do not write.)

2. “Grief” describes a pattern which recurs, with different variations, in the tapestry of life [*Lebensteppich*]. If a man’s bodily expression of sorrow and of joy alternated, say with the ticking of a clock, here we would not have the characteristic course of the pattern of sorrow or of the pattern of joy.

We now reach the final occurrences of *Lebensform*, in PPF. These remarks, loose and unpolished, were penned in 1948-9, much later than the versions of the PI we have been looking at here. Yet Wittgenstein’s preoccupation with Turing, and with embedding the notion of *Lebensform* deeply into the foundations of mathematics, and of logic itself, continued. In 1947, as we have seen, he was still writing in notebooks about the distinctive “Do

What You Do” machine in Turing’s paper of 1936/7, and stressing that Turing’s “machines” are really *humans* who calculate.

Beginning in the *Blue* and the *Brown Book*, language-games were conceived anthropologically: the aura of everyday life and culture were being drawn in.¹⁰⁷ Here “common sense” and “ordinary phraseology” with signs and words are understood to be at work in the backdrop, they are part of the “reality”.¹⁰⁸

But Wittgenstein also described his language-games as “experiments”.¹⁰⁹ There are dangers in speaking only of tribes: for what was coming to matter most to Wittgenstein’s pursuit of the logical is that these language games are *possible* formations of action, not relativized meanings.¹¹⁰ Wittgenstein’s mature uses of “language games” in PI – with their stress on snipping and transporting and amalgamating and snapshotting the living variety and interlocking applications of procedures in language, imagined “forms of life” (and the “criteria” embedded in these, and so on) – replace or reconstrue the notion of “common sense” with something more plastic, simpler, more evolutionary and dynamic. They are more ubiquitous, flexible and creative: more logically self-critical and self-standing.

The extraction from animal life involved in Wittgenstein’s above-quoted remarks from PPF i further develops the idea of embedding *Lebensformen* in life. Forms of life and imagined language games, as we have said, bring *life* to possibilities and necessities, characterizations of characterization itself, and they implement repeatable possible procedures or models or routines. What the dog does not have – so far – is the power to embed those far-reaching routines that imbricate one another in our human lives. The dog may anticipate its master’s needs, anticipate, remember

¹⁰⁷ Cf. Engelmann 2013b for a helpful analysis of the transitions.

¹⁰⁸ On the phrase “phraseology” see Floyd 2013a, forthcoming c; it may be found in Russell 1920, pp. 141, 175, 192 and occurs in BB, 69; it also appears in Wittgenstein’s manuscripts and lectures around differing conceptions of numbers and mathematics, cf. e.g. MS 121, 76 (1939), MS 126, 131 (1942-43); MS 127, 194 (1943) and LFM 18, 91, 98. Floyd and Mühlhölzer (forthcoming) discuss this notion at length in the context of an analysis of Wittgenstein’s annotations to a 1941 edition Hardy’s *Course of Pure Mathematics*.

¹⁰⁹ BB, 42, 47; This phrasing is also used in the Francis Skinner version of the *Brown Book*, now at the Wren Library, Trinity College Cambridge.

¹¹⁰ Cf. Engelmann 2013b 208ff. on the danger of “tribes”, noted by Wittgenstein himself.

how to apply words to things. But without the farther-reaching “modifications” of our “complicated” *Lebensform*, it cannot hope that its master will arrive at a quarter past three in the afternoon two days’ hence, vowing revenge by three-thirty if not. Without the human signature, the capacity to write down chains of inferences, to grasp operations and routines that interweave in this way, to project these into new contexts in shareable routines, without all these specific modifications of a life that contains expectations, such is not part of the life form of the dog.

In PI Wittgenstein’s language-games model issues of difficulties that lead to genuine puzzlement: contrasts and comparisons among routines, confusion about how to shuffle parameters, shifts in aspects of uses of signs, of different routines, in our *perception* of possibilities and necessities. They draw out Wittgenstein’s whole idea of scaffolding: we can put a certain procedural cluster of activities up, climb upon it, and then take it down, using it to build whatever *particular* structure we like. Scaffolding is supported by itself, and supports nothing. Its parts substitute in for one another step by step, or not (scaffolding is a kind of chaining, or binding, after all). We leave it behind when we move to forward a particular structure, a particular thing said – when the structure we seek is erected. But its signature, its way of allowing us to frame and draw out new aspects, is there to *be* drawn out, unfolded, re-erected when we need it.

The tapestry of life [*Lebensteppich*] is colorful, something we walk on and can be enchanted by. It is a suitably human image. The foundations are woven, but hold, have their integrity within structures we build, or even just upon the given ground, without either absolute or merely relative simples. This is a complement to the colored squares Wittgenstein earlier devised to communicate his new view of simples, quoting from the *Theaetetus* (PI 46), but it draws the image out dynamically, embedding it. Looking down on this profusion of interwovenness, we may see aspects, patterns, trace routines, see them break off, focus on a part, or try to take in the whole. The whole may have borders, but may always be added onto. Wittgenstein’s idea of “criss-cross philosophy”, of weaving, is

apposite.¹¹¹ Logical necessities rest upon no metaphysical foundations of the whole, no “glue”¹¹², no prior ordering or features of the world, but instead upon our activities in fashioning links in our world, one by one. We are binders, working against a universal backdrop, a whirl of life. Logic is concerned with establishing procedures and possible procedures, as well as plumbing the limits of certain kinds of procedures. Insofar it can come to be indefinitely imbricated and impressed, ubiquitously, in lives with speech and with language.

Wittgenstein’s explicit reconsideration of logic as “sublime” reaches a certain high watermark in PI 92, 122. Here, meditating on his *Tractatus* idea of a “final analysis”, in which the “essence” of language, propositions, and thought-as-such lies hidden “*beneath*” the surface, he describes his contrasting, more mature idea as one according to which the very depth of the possibilities of these phenomena are something

...that already lies open to view, and that becomes *surveyable* [*übersichtlich*] through procedures of ordering [*durch Ordnen*]... (PI 92)¹¹³

And he adds that

A surveyable representation [*Übersichtliche Darstellung*] produces precisely that kind of understanding which consists in ‘seeing connections’. Hence the importance of finding and inventing *intermediate links* (*Zwischengliedern*). (PI 122)

It is notable, as I have said (though it is hardly often stressed) that the ideas of sequencing, ordering, finding and inventing

¹¹¹ Z 447, discussed also by Diamond 2004. In Diamond’s sense, I am arguing that in a sense the PI still addresses itself to a Big Question (“What is the nature of the logical?”). But the mature Wittgenstein answers that one can only answer *that* question in a certain down to earth way, by way of smaller pieces and reflection on their interweavings in life. The transition as I am envisioning it is less from “wholesale” to “retail”, giving up on Big Questions altogether (as Diamond has it), and more about projecting the Big Question all the way through to its real foundations. For the origin of “crosswise” proof, which I think is the origin of this later remark, see BT 130, 456: “The inductive proof puts the equation together crosswise, as it were, instead of lengthwise.”

¹¹² As Wittgenstein put it in his letter to C.K. Ogden about *Tractatus* 2.03 (CCO 1973, 23). Cf RFM V 24 for a critique of another kind of glue-conception that causes us to lose hold of the special character of logical technique when we mathematicize logic.

¹¹³ I have slightly altered the translation here, despite the reasonable argument of the editors of PI. The manuscript basis here (157b, 14-15) shows some struggling on Wittgenstein’s part to formulate the idea with sufficient precision.

intermediate links all belong to Wittgenstein's particular conception of deductive logic, and so, of analysis. A deductive argument is classically regarded as a chain among terms and arguments, an ordering of thoughts or propositions, as we have seen. Only more recently has the process of logicizing come to be construed in terms of sequences, and, most recently of all, as a sequence of step-by-step commands, or orders, a routine off-loaded and treated as a transportable module.

Behind these remarks in PI lie many others. In manuscripts of 1939-40, written during and after Turing's attendance at his lectures on the foundations of mathematics (LFM) and later published as RFM III, Wittgenstein had attempted to frame a deepened, more refined application of his perspective on logic and the foundations of mathematics, returning to the themes broached in FF. In these remarks he developed and applied his notion of *Übersichtlichkeit* to *Principia Mathematica*, criticizing Whitehead and Russell's version of a *reduction* of arithmetic to logic.

His examination of the *surveyability* of proof swims in the wake of his discussions with Turing in the spring of 1939, and points forward to what we see in PI. As Felix Mühlhölzer makes clear in his commentary on these manuscripts in RFM III (2006, 2010), to be "surveyable" simply means to be something that can be "copied, in the manner of a picture". The image is one of a shareable survey, one we can express and work through, and reproduce, sharing it with harmonies and agreements among us about what counts as a sufficient copy. Wittgenstein is working his way forward toward the applications of *Lebensform* in PPF.

The idea is that the so-called "foundations" of mathematics should be *worked through* or *ordered* in the light of Wittgenstein's conception of logic. But in his writings he is offering, as Mühlhölzer explains, a *Klarlegung* or *Grundlegung* of mathematics, not a *Grundlagen* (cf. LFM lecture 1, Mühlhölzer 2010 *passim*). As Turing himself wrote (1944/5, 245), mathematical logic is "an alarming mouthful for most mathematicians, and the logicians are not very much interested in making it more palatable". The solution Turing proposes is to make symbolic logic disappear: he proposes to rework and organize the ordinary "phraseology" of "the

mathematician-in-the-street” so that whatever merging of logic and mathematics might go on, it will require the mathematician to learn, ideally, very little of mathematical logic.

So it is today, with the use of modern computational routines: software is developed in light of our uses of it, and to achieve further compression, routinization and also, of course, freeing of humans *from* calculative routines.

4.6 PPF xi 343-363: *The Ribbon of Life* (Band des Lebens), Lebensform as the “Given”

PPF xi:

343. But am I really trying to say that the certainty of mathematics is based on the reliability of ink and paper? No. (That would be a vicious circle.) – I have not said *why* mathematicians do not quarrel, but only *that* they do not.

344. It is no doubt true that one could not calculate with certain sorts of paper and ink, if, that is, they were subject to certain strange alterations – but still, that they changed could in turn be ascertained only through memory and comparison with other means of calculation. And how, in turn, are these tested?

345. What is the accepted, the given, is – one might say – *forms of life*.

...

358. One can indeed be convinced by the evidence that someone is in such-and-such a state of mind: that, for instance, he is not pretending. But there is also ‘imponderable’ evidence here.

359. The question is: what does imponderable evidence *accomplish*? Suppose there were imponderable evidence for the chemical (internal) structure of a substance; still, it would have to prove itself to be evidence by certain consequences which *are* ponderable.

(Imponderable evidence might convince someone that a picture was a genuine . . . But this *may* be proved right by documentation as well.)

360. Imponderable evidence includes subtleties of glance, of gesture, of tone.

I may recognize a genuine loving look, distinguish it from a pretended one (and here there can, of course, be a ‘ponderable’ confirmation of my judgement). But I may be quite incapable of describing the

difference. And this not because the languages I know have no words for it. Why don't I simply introduce new words? – If I were a very talented painter, I might conceivably represent the genuine and the dissembled glance in pictures.

361. Ask yourself: How does a man learn to get an 'eye' for something? And how can this eye be used?

362. Pretending to be in pain, for example, is, of course, only a special case of someone producing expressions of pain without being in pain. If this is possible at all, why should it always be pretending that is taking place – this very special pattern in the weave of our lives [*auf dem Band des Lebens*]?

363. A child has much to learn before it can pretend. (A dog can't be a hypocrite, but neither can it be sincere.)

In the second, rougher, part of PI we see that the accepted, the “given” are *Lebensformen* (PPF xi, s. 345). Here Wittgenstein is pointing toward issues thematized in *On Certainty*, drawing out the notion of “certainty” in mathematics. Forms of life are just wherever we do begin. There are hard necessities in procedures, step-by-step. There are simples. But it is always contestable to begin here rather than there, to break off one routine and turn to another, to take a part of a routine out and put it in elsewhere, to change a routine, alter one's sense of a face, to take a face in, and so on. Forms of life evince aspects that may be altered, compared, worked with, and run through.

As part of logic in a primordial sense, this idea of the “givens” is very ancient. Euclid, as we have said, calls them “data”.¹¹⁴ As we know, this just means something given (say, a particular construction) with a request for us to show that something else *can* be done given it. This sort of “given” is not, as it became in Russell later on, an empiricist's object of immediate and infallible “acquaintance”: that image breaks the chaining idea, pinning it down, limiting it. Rather, in later Wittgenstein a “given” is a gadget, a starting point in life, something cobbled together *in* the stream of life.

¹¹⁴ And so, appropriately, Kenny (2006, 13) calls “forms of life” “the ultimate data” in W, a replacement for the simple “atoms” of the TLP.

Forms of life, as “the given” are the simples, our starting points. They are not given through self-evident propositions, such as those put forward as such, dogmatically, by Moore. Nor are they proposed as actual courses of action or descriptions of what is really going on when we speak – though as Wittgenstein repeats again in this context (just as he had in PI 240), mathematicians in fact do not come to blows over what is to count as an acceptable step (PPF xi 343). When once they did – intuitionists and classical logicians – as a matter of fact they found a way around it, in procedures of formalization, with the help of logic. Forms (possibilities of structure) are whatever we make to *be* “given” in the course of considering and exploring, not only proposed objects of comparison, but proposed starting and juncture and stopping points, simples, accepteds (as we might call them). It does not matter, for purposes of the classification of logic itself, *where* we begin. What matters is that we get the beginnings themselves into the right sort of frame.

The ancient metaphysical picture of Aristotle, that relatively static tree structure of concepts, does not hold for the general foundation of logic itself. But in Turing and Wittgenstein, logic *itself* shows its own limits. Not by way of any “miracle”, but by way of *us*.

Wittgenstein’s image of a “band” or “ribbon” or “volume” of life [*Band des Lebens*] is his answer to the whole idea of a colorful conception of life, perhaps even *his* book’s life, perhaps his own life’s work in philosophy (in logic). It is a reworking of the ancient idea of chains and bindings in logic. It is an image of, for, in and by logic as philosophy.

We are *binders*: of words and sentences, of procedures, of routines and activities, of ourselves with one with another, of thoughts and words and actions. We put elements together in arrangements that hold. We are capable of binding ourselves to procedures and routines. Chains and bindings can be broken, or broken off, or amalgamated, or simplified, or rearranged, shuffled, entwined, according to purpose – or, in fact, merely aimlessly, emptily, or decoratively. In Wittgenstein’s mature philosophy, these bindings must be fashioned without a base-level of fixed simplicity,

even though every binding entangles us with simples. In PI at PPF xi 358ff. this idea of a *Band des Lebens* works out the whole idea of *pretense*, of lies, of what logic makes us capable of, and how. This is no accident. Wittgenstein is grasping at getting down to the very idea of dealing with possibilities and necessities of human actions with truth as such. He reverts to *his* reconception of the notion of acquaintance, his Master Simile.

This brings us to the notion of the “I”, the subject. In the *Tractatus* it had disappeared, a kind of formal “limit”, inconceivable as any kind of substance or relata, on the ground that this *would* have made it a complex, but instead, a soul or subject would have to be *simple* (TLP 5.5421). In Wittgenstein’s mature philosophy the thinking “I” is still not an object. But it is no longer conceived of as a limit, a well-behaved relational aspect or approximation of criterial or grammatical procedures. Instead, its simplicity is also embedded in the fluidity, the interactional situatedness, of voice, procedure, and expression.¹¹⁵ It is not to be conceived of as either continuous *or* “gappy” – as if it were something that might be completed by cuts and collections, like the continuum was by Dedekind.¹¹⁶ Instead, it is formed, *bound* and *arranged*: bound by necessities and possibilities, including those of writing, those of a human body and bodies, of traditions and daily routines and voices. It is bound by its own bindings (of words and thoughts).

Wittgenstein pursues a one-leveled, multi-aspectual view of form, not ascending to a meta-level or supposing that form may be reduced, Hilbert-style, to strings or signs or an axiomatic

¹¹⁵ See Sluga forthcoming for an interesting discussion of the evolution of Wittgenstein’s thoughts about the “I” through the period of *The Blue Book*.

¹¹⁶ Laugier (2005, 68) pointed me toward this passage at Z 648: “One language-game analogous to a fragment of another. One space projected into a limited extent of another. A ‘gappy’ (*löchriger*) space. (For ‘inner and outer’.)” I would translate “*löchriger*” differently than Anscombe does. The quotation marks clearly signal an allusion to someone else’s use of the word. My sense is that Wittgenstein is critically alluding to the picture of using Dedekind cuts to fill in “gaps”, as treated in Russell 1920. Dedekind himself strikes me as more careful with the image, especially in relation to Euclid (see Dedekind 1888/1996, Preface to the 1st ed., 793; compare his 1872/1996 §3, 771; these are discussed in Floyd 2013b). In Wittgenstein’s work on real numbers, he *always* attacks this picture of gaps, on which see Floyd and Mühlhölzer manuscript. He is considering transposing his way of thinking about the foundations of mathematics into the problem of inner and outer here. Note that a chain *has* no inside and no outside, except when it is folded up, hidden through its arrangement.

formulation of a theory, or lodged, Hardy-style, in actual, albeit purely metaphysical conceptual structures (propositions) intrinsically true or false. This is his difficult road in recasting the logical: it relies throughout on analogical, comparative thinking. And it cuts very much against the usual grain of discussions of intension and extension, concept and object, in traditional logic, because it is analyzing those notions in terms of something else, taken as given: aspects and *Lebensformen*.

The contrast is clear. If one treats logic as “merely formal” in Hilbert’s way, or the “formalist” way that Frege was concerned to attack, then one is *forced* to say that there must be “metallogic”, something “outside” the formalism.¹¹⁷ Wittgenstein always attacked this idea.

The huge shift from Aristotle’s notion of form is that in Wittgenstein forms do not ultimately have to correspond to any fixed tree-like taxonomic structure or ordered system of life forms, essences, substances, or concepts except relatively, and revisably, dynamically, as with chains and chains of chains. No such hierarchical and grounded structure of forms is revealed in chains of reasoning. Forms are linked in procedures, not ranked by terms. We cannot any longer regard, e.g., particular syllogisms as something revealing life-form essences and their (intensional and extensional) relations. In Wittgenstein a logical train of thought or an analysis may compel us, but it need not tell us, either *that* or *why* – again, unlike in Aristotle. We begin with such a chain, and we end with it. Logic does not represent: in it we propose, bind, and dispose. It reveals itself *in* that, and that alone.

Forms – logical and living ones – are linked, not ranked. In Wittgenstein forms have to do with procedures, routines, possibilities and necessities thereof – these are so ubiquitous. And so he self-inscribes and self-thematizes the forms of his own procedures in the PI. His remarks are deeply self-embedded in if-thinking, in *possibilities* of procedures or phenomena, in loops and pieces of proposed and explored chains of argumentation. The same might be said of the *Tractatus*, though there the idioms are less

¹¹⁷ Even Schopenhauer falls into talk about “metalogical truths” of logic (2010, 56).

modular, because Wittgenstein's ideal of simplicity is one of groundedness: as he later put it – critically – he had composed a “calculus of undefinables”.¹¹⁸ This, as Turing demonstrates, we cannot have in *general*.

In later Wittgenstein the modes (in PI “modifications of this complicated form of life” (PPF i 1)), are explicitly thematized as *aspects*, are contextualized and emerge from structure and comparison, being treated occasionally compositionally, bottom-up. Wittgenstein is inclined more and more, as time goes on, to stress the varieties of ways aspects emerge and strike us, and the variety of our own activities in binding up chains of thoughts. He explores the differences here between our capacity as binders and those of other life forms (dogs and rabbits), but what is at stake here are *our* bindings, our concepts (“natural history” in Wittgenstein's sense). *Aspect* is a logical, not a psychological notion.

The metaphor of a chain naturally connects, then, not only to “form” but to “life”. The great “chain” of Being is no longer, as in Lovejoy, an ordered, gap-free taxonomic structure in which all possibilities are eventually realized. It is a series of processes and fluid though structured events of formations of formations of (and in) life, necessities embedded in a world of contingencies, in what we can do and do and fashion, and what follows from that about possibilities for other forms. Such chain-procedures or possibilities, which we follow and put together and share, afford opportunities for exercises in characterization, analysis, and simplicity that are themselves designed to characterize, analyze and simplify the notions of characterization, analysis, and simplification. In the later, mature Wittgenstein chains have no absolute end or absolute beginning, in terms of a fixed place for measuring: there remains *in general* only the schematic “it begins” and “it ends”, the possibility of binding (with words, thoughts, actions, streets, gestures, procedures...) as such.¹¹⁹

There is neither an absolute, nor a merely relative notion of simplicity here, and no doctrine at all of *undefinability*. There is

¹¹⁸ Cf. MS 111, p. 31 (1931). Wittgenstein mentions Plato in this passage *en passant*, anticipating his later invocation of the *Theaetetus* at PI 46.

¹¹⁹ The resonance with Turing's work on morphogenesis 1952 is striking.

simply the ubiquity of the capacity for the binding. It is a vision of the sublimating of logic – *domesticated*.

This may sound like an evolutionary picture in Darwin's sense, and the affinities between Wittgenstein and the pragmatists, for whom Darwin was a central figure, have often been noted.¹²⁰ While he would not have denied that logic has evolved from somewhere, historically, Wittgenstein would also have insisted – every bit as much as the pragmatists, in their anti-reductionist moments – that we *make* history, as actors *within* it, and especially by talking with one another.¹²¹

The words, on this picture, with their evolutions of patternings and usages, are part of our dynamic reality. They do not, of course, *constitute* it alone: Wittgenstein is no linguistic idealist. Unlike Hegel, Wittgenstein does not suppose that there is any particular end of history to be expected: Why would he? If history has ended, if an ultimate analysis of logical analysis itself (both *logos* and *logistike*)¹²² has been gotten to, then this has happened by embedding it in everyday life, ubiquitously. Then it is with schematic, open-ended, but ever-possibly-broken-off ways of thinking that we must live and think.

As I have interpreted PI, Wittgenstein was quite aware that he wrote in a post-Aristotelian world, in which perception would not suffice for any automatic or natural sharing of form unaided by artificial forms of articulation. He lived during a time in which the form/content distinction itself was being radically recast as an ideal of the systematicity of judgment. An overarching systematicity of possibilities and/or actualities could no longer be accepted *a priori*, even as an ideal, as a collectivity of form, ordered into a given, ranked whole. This much incompleteness does tell us.

Forms are linked, not ranked, in chains. Chains are put together, arranged, and then hold fast, articulated. Chains may be linked to one another, side-by-side or end to end or interwoven as in a coat of mail. Chains may be folded and unfolded (the root of the idea of a conclusion being *implicit in*, unfolded from, the premises), their

¹²⁰ Cf. e.g., Misak 2016.

¹²¹ Cf. Wittgenstein 1979 (WVC), 34, n. 1.

¹²² On this distinction see Stein 1988, discussed in Floyd 2013b.

links substituted for one another indefinitely. Their links may be broken off, reassigned, moved, amalgamated, broken into further intermediate links, and so on. Some chains are made of paper, some of wool, some of iron. Some are linear, some are tree-like, some are themselves closed loops. In general, the “being with in a chain”, like the notion of “being with in a queue”, is something we do *with*, not *beside* or *to* others. Chains may be reproduced, copied in the manner of pictures: they are, or can be made, *übersichtlich*.

We are responsible for fashioning the links *in* logic, even in the *Tractatus*’s philosophy. There it is things or objects (*Gegenstände*) that are said to hang together in the *Sachverhalte* as in a chain, and names are said to be linked in an elementary sentence as in a chaining [*Verkettung*].¹²³ These chainings open up, as they picture, possibilities, opening and closing them off. Placed in operational, comparative settings, they thereby allow us to draw in questions about how one such opening may or may not be an opening or closing of another, how one proposition may or may not be logically related to another.

But only in the later Wittgenstein does *how* we link, what we specifically do *in* linking, become a thoroughgoing, integrated part of logic, part of life itself, rather than (as in the *Tractatus*), something irrelevant to what is bound to show forth anyway, no matter what we do or say or try to picture or imagine or fashion or how we live.¹²⁴ And yet, for all that, logic has regained its self-standing, ubiquitous character in the later philosophy.

Early and late for Wittgenstein, *in* logic (philosophy) our linkings are part of reality, and our notion of reality, as such. They can be – must be – broken off, returned to, rearranged, and broken off again. Logic does not represent the world as it is. It composes it as it may and must be. As the song has it: forty-nine reasons, all in a line, all of them good ones (bound with the links of logic), all of them lies.¹²⁵

We are *binders*. We bind ourselves to one another and to procedures. We band into groups, friendships, cities. We band

¹²³ TLP 2.03, 4.22.

¹²⁴ Narboux 2014, Floyd forthcoming a.

¹²⁵ “Forty-Nine Bye Byes”, on Crosby, Stills and Nash Greatest Hits 2005.

together words, arguments, volumes, stories, lies, ponytails, tapestries, costumes, dress. We band into musical groups, peoples, fans. We band through the binding of remarks and reminders. The regressive culmination of Wittgenstein's remarks on *Lebensformen* in the *Investigations* occur in PPF – what I still think of as “Part II”. Here the notion of a “carpet” or “tapestry” of life [*Lebensteppich*] (PPF i, 2) is developed into that of a *Band des Lebens* (PPF xi 362). This is the image of a parade of life forms, a colorful ordering of life, lifted from a romantic tradition in which life itself is regarded as a ribbon. The passage is translated by Hacker and Schulte as a “weave” of life¹²⁶ to pick up on the connection with the image of *Lebensteppich*; Anscombe had “pattern in the weave of our lives”, attempting to be a bit broader. But *Band*, more or less equivalent to “band” in English, is broader still, and more pertinent to Wittgenstein's quarry with his idea of *Lebensform* as showing us what logic is.^{127, 128}

Abbreviations of Wittgenstein's Works:

AWL	Wittgenstein 1979b
BB	Wittgenstein 1965
BF	<i>Bearbeitete Frühfassung</i> PI in Wittgenstein 2001
BT	Wittgenstein 2005
CCO	Wittgenstein, Ogden, Ramsey 1973
EPB	Wittgenstein 1969
FF	<i>Frühfassung</i> PI in Wittgenstein 2001

¹²⁶ Stern 2004, 167 quotes RPP II §§624-9, which use the interesting notion of a “filagree”.

¹²⁷ I profited from discussion of a draft of this paper with the audience at the May 2016 “Forms of Our Life with Language” Conference, Ludwig-Maximilians-Universität, Munich supported by the Carl Friedrich von Siemens Stiftung and organized by Chritian Martin. For work on sources I have relied on Biggs and Pichler 1993, Wittgenstein 1999, 2001, 2015, and von Wedelstaedt 2007, to which Felix Mühlhölzer called my attention. I am indebted to him for many helpful discussions, as well as Mauro Engelmann, Martin Gustafsson, Akihiro Kanamori, Yrsa Neuman, Alois Pichler, Sanford Shieh, Jonathan Smith, Max Weiss, and the students in my spring 2016 Seminar on Wittgenstein at Boston University, especially Kurt Blaenkshen.

¹²⁸ Some corrections including rewordings added to this version Dec. 17, 2016 and some references amended Jan. 25, 2017 and June 29, 2017 (first publication Dec. 16, 2016), by editor.

LFM	Wittgenstein 1976/1989
NB	Wittgenstein 1979a
OC	Wittgenstein 1998
PG	Wittgenstein 1974
PI	Wittgenstein 2009
PPF	In Wittgenstein 2009
PR	Wittgenstein 1980a
PR	Wittgenstein 1989a
PT	In Wittgenstein 1989
RFM	Wittgenstein 1978
RPP I	Wittgenstein 1980b
RPP II	Wittgenstein 1980c
UF	In Wittgenstein 2001
WA	Wittgenstein 1993-1997
WC	McGuinness (ed.) 2008
WCLM	Stern, Rogers and Citron (eds.) Forthcoming
WVC	Wittgenstein and Waismann 1979
Z	Wittgenstein 1981
ZF	<i>Zwischenfassung</i> PI in Wittgenstein 2001

References to Wittgenstein's Nachlass 1999 use the reference system first broached in von Wright 1969.

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